Favor-trading with Incomplete Information

Simo J. Kalla

Abstract

We investigate whether and how individuals who stand to gain from favor-trading can best form cooperative relationships in an environment with private information about each agent's ability and willingness to do favors. For agents with a low discount factor (low types) cooperation is not incentive compatible, for agents with a high discount factor (high types) it is. Both types receive privately observed opportunities to do favors with positive probability each period. We show high types are always able to separate from low types. Separation is implementable as soon as a high type receives a favor opportunity if the opportunities are independent across agents. If they are mutually exclusive, high types continue to separate with probability one if one of the agents is designated to do the first favor and that agent is a high type. Equilibria that designate an agent to act first implement separation approximately twice as slowly as symmetric equilibria that task the first high type with opportunity to separate first. Therefore the latter type of symmetric equilibria may dominate the former type of non-symmetric equilibria.

1 Introduction

This paper studies whether and how individuals who stand to gain from trading favors can best form cooperative relationships in an environment with private information about each agent's ability and willingness to do favors. Previous models in the favor-trading literature focused on optimizing favor-trading relationships under complete information. This paper introduces incomplete information about player types. The central question addressed is whether cooperation can be maintained in favor-trading relationships after the introduction of non-cooperative players into the pool of potential trading partners, and if so how can the cooperative players separate themselves most efficiently from the non-cooperative types.

For the purposes of this dissertation favor-trading is considered to be non-monetary trade in goods, services or opportunities and favors are assumed to be efficient. The model contains a positive measure of players with a low discount factor (low types) who do not find cooperation beneficial, and a positive measure of players with a high discount factor (high types) who do. Players receive opportunities to do favors for each other (favor opportunities) according to either a mutually exclusive or independent distribution, but these opportunities are private information.

As an example, consider a firm with several parallel divisions that function independently under separate managers. Suppose two new managers have been recruited to head the marketing and finance divisions, respectively. Each manager's job is to maximize productivity within her own division, but every once in a while one of the managers receives a new idea or opportunity that would be beneficial for her division but even more beneficial for the neighboring division. Monetary side payments are not allowed, but reciprocation in similar favors can provide a basis for mutual gains if both managers are sufficiently patient. However, the managers do not know each other's discount factor, which in this example could be interpreted as the likelihood of staying with the firm long-term. So how should the managers proceed?

The main result in this paper is that the high type players are almost always able to separate themselves from the low type players at the first available opportunity by using an "equality matching" (EM) mechanism if opportunities to do favors arrive independently. EM simply means that each agent waits for reciprocation of a previous favor before doing the next one. In the case of mutually exclusive favor opportunities, separation is still guaranteed for high types with probability one if one of the players is designated to do the first favor, and separation will occur as soon as the designated player receives a favor opportunity, assuming she is a high type.

However, such strategies induce separation only half as quickly (roughly speaking) as strategies that call for the first player to receive a favor opportunity to do the first favor, and hence separate if she is a high type. In equilibria based on such symmetric strategies, separation is only guaranteed with probability one under independent favor opportunities, but not under mutually exclusive favor opportunities. The paper establishes a bound on the number of periods in the mutually exclusive favor opportunity case during which the low types will not mimic the high types with positive probability. An important consequence of this result is that more information (mutually exclusive favor opportunities) leads to a worse outcome.

The paper also compares the equilibria involving a designated first favor maker (DFFM equilibria) to equilibria characterized by symmetric strategies (SS equilibria) and finds that either may dominate depending on the parameter values. Numerical results suggest that SS equilibria dominate DFFM equilibria in cases involving relatively impatient agents likely to receive favor opportunities early on, whereas DFFM equilibria dominate when agents are very patient, but the probability of receiving favor opportunities is low. To see why, consider the case of mutually exclusive favor opportunities and suppose the probability of receiving an opportunity to do a favor is approximately one half. Then it is almost certain that one of the two agents will receive a favor opportunity during the first period, and therefore under a symmetric strategy for signaling type, separation will almost certainly occur during the first period, which is all the more important for relatively impatient high type players eager to gain from the benefits of an EM endgame. Designating one of the two players to do the first favor would halve the chance of separation in the first period, which is all the more costly when the high type players are impatient to begin a cooperative relationship.

The rest of the paper is organized as follows: Section 2 describes the model and our equilibrium concepts including how key concepts from AB [1] as they translate to our streamlined version of their model and how other favor-trading literature relates to this paper. In section 3 we analyze the case of a patient agent facing an unknown type, and investigate how the two can best form a favor-trading relationship. In section 4 we analyze the case of two unknown type agents when opportunities to do favors are mutually exclusive. In this section our focus is on the existence of separating equilibria, and how best to separate into the case analyzed in one-sided incomplete information case analyzed in the previous section. In section 5 we extend the analysis to the independent favor opportunity case, and compare the results to the case of mutually exclusive opportunities. Section 6 concludes. An appendix follows and references are at the end.

2 The Model

Consider the earlier motivating example: A firm has several parallel divisions that function independently under separate managers. Suppose two new managers have been recruited to head the marketing and finance divisions, respectively. Each manager's job is to maximize productivity within her own division, but every once in a while one of the managers receives an opportunity to help the other division at a cost to her own. The ability or opportunity to help is private information, but when possible the cost is known to be less than the benefit. Monetary side payments are not allowed, but reciprocation in similar favors can provide a basis for mutual gains if both managers are sufficiently patient. But the managers do not know how patient the other is, or how likely she is to stay with the firm long-term. To address whether and how they can form a cooperative relationship we analyze the following formal model.

Two agents, a and b, are randomly picked from a population with $\mu_o \in (0,1)$ of high types with discount factor δ^H and $1-\mu_o$ of low types with discount factor δ^L. Each agent has utility function $u(x)=x$. They play an infinitely repeated stage game with the following structure. At the beginning of each period nature allocates an opportunity to do a favor (*favor opportunity*) according to either a mutually exclusive or an independent distribution. Under a mutually exclusive distribution either agent a or b receives a favor opportunity with equal probability, $p \in (0,1/2)$, or neither does with probability $1-2p$. Under the independent distribution each agent receives a favor opportunity with probability $p \in (0,1)$. Favor opportunities are private information. An agent who receives a favor opportunity may either keep it private and incur no cost, or do a full or partial favor of size $x, y \in (0,1]$, for agents a and b, respectively, at a cost equal to the size. The benefit to the recipient is ky or kx, for agents a and b, respectively, where $k>1$. For example, if agent a does a favor of size x, flow payoffs to (a,b) are $(1-x, kx)$. Favors, including their size, are public information. The stage game is repeated in each subsequent period.

To see how favor-trading works consider the following game called *equality matching (EM).* In EM of level $z \in (0,1]$, one agent is called *advantaged,* the other *disadvantaged.* The disadvantaged agent is said to owe the advantaged agent a favor of size z. If the disadvantaged agent does a favor of size z, she becomes advantaged and the other disadvantaged. If she does no favor, she remains disadvantaged. Favors of size other than z are not part of equilibrium play and can be deterred by Nash reversion. When $z=1$, the game is called *full equality matching.*

For the moment, consider a game of full equality matching between two high types in a complete information environment. Suppose agent a is disadvantaged, b advantaged. Let $(\underline{u}_{em}, \bar{u}_{em})$ denote the average discounted payoffs expected by agents (a,b), or more generally by disadvantaged and advantaged agents, respectively. Let $\sigma_{em}(\underline{u}_{em}, \bar{u}_{em}) = (\sigma^a_{em}(\underline{u}_{em}, \bar{u}_{em}), \sigma^b_{em}(\underline{u}_{em}, \bar{u}_{em}))$ denote the EM strategy profile that implements the payoff pair $(\underline{u}_{em}, \bar{u}_{em})$. Under σ_{em} the payoffs are

$$\begin{aligned}
\underline{u}_{em} &= p\delta^H \bar{u}_{em} + (1-p)\,\delta^H \underline{u}_{em}, && (1)\\
\bar{u}_{em} &= p\left(1-\delta^H+\delta^H\bar{u}_{em}\right) + p\left(\left(1-\delta^H\right)k + \delta^H \underline{u}_{em}\right) + (1-2p)\,\delta^H \bar{u}_{em}. && (2)
\end{aligned}$$

The first equation consists of two events: (i) with probability p agent a receives a favor opportunity, does a full favor $(x=1)$, and becomes the advantaged agent; that is, agent a is promised continuation payoff $\bar{u}_{em}$, (ii) with probability $(1-p)$ agent a receives no favor opportunity so her flow payoff is zero and her continuation promise remains $\underline{u}_{em}$ along with her disadvantaged status. The equation for payoff $\bar{u}_{em}$ consists of three events that occur with probabilities p, p and $(1-2p)$, respectively: (i) agent b receives a favor opportunity, does no favor and receives a flow payoff of 1 instead, and her continuation promise remains $\bar{u}_{em}$ as she is still advantaged, (ii) agent a receives a favor opportunity, does a full favor $(x=1)$ so agent b receives a flow payoff of k, but her continuation payoff drops to $\underline{u}_{em}$ because she now owes agent a the next favor, and (iii) neither agent receives a favor opportunity, so agent b's flow payoff is zero and her continuation payoff remains $\bar{u}_{em}$. The two previous equations contain two unknowns, $\underline{u}_{em}$ and $\bar{u}_{em}$, solving for these yields

$$\underline{u}_{em} = \frac{\delta^H p^2 (1+k)}{1-\delta^H (1-2p)}, \tag{3}$$

$$\bar{u}_{em} = \frac{p\left(1-\delta^{H}(1-p)\right)(1+k)}{1-\delta^{H}(1-2p)}. \tag{4}$$

For the simple EM strategy profile to be a *Nash equilibrium* (NE) in each stage game, neither agent can have a profitable deviation available to them. It is trivial that the advantaged agent has no profitable deviations as she just waits for reciprocation, but does no favors. Public (observable) off-equilibrium path deviations, such as the advantaged agent doing a favor or one of the agents doing the wrong size favor, can easily be deterred by threat of autarky (no more favors). Therefore, we only need to check that it is not profitable for the disadvantaged agent to do no favor despite receiving a favor opportunity. As usual, it is enough to consider a one-shot deviation. Agent a's discount factor has to be high enough that the incentive compatibility constraint below is satisfied.

$$\begin{aligned} ICC^{a}_{em} : \delta^{H}\bar{u}_{em} &\geq 1-\delta^{H}+\delta^{H}\underline{u}_{em} \\ \Longleftrightarrow \bar{u}_{em}-\underline{u}_{em} &\geq \left(1-\delta^{H}\right)/\delta^{H}. \end{aligned}$$

Using equations (3) and (4), ICC^{a}_{em} may be written as

$$\begin{aligned} \tfrac{p\left(1-\delta^{H}(1-p)\right)(1+k)}{1-\delta^{H}(1-2p)} - \tfrac{\delta^{H}p^{2}(1+k)}{1-\delta^{H}(1-2p)} - \tfrac{1-\delta^{H}}{\delta^{H}} &\geq 0 \\ \Longleftrightarrow \tfrac{1-\delta^{H}}{\delta^{H}\left(1-\delta^{H}(1-2p)\right)}\left(\delta^{H}p(k-1)-\left(1-\delta^{H}\right)\right) &\geq 0. \end{aligned}$$

It follows $\delta^{H} \in (0,1)$ must satisfy $\delta^{H}p(k-1)-\left(1-\delta^{H}\right) \geq 0$ for ICC^{a}_{em} to hold. Solving $\delta^{H}p(k-1)-\left(1-\delta^{H}\right) \geq 0$ for δ^{H} yields $\delta^{H} \geq \frac{1}{1+p(k-1)} = \delta^{*}$. We use this boundary discount factor to define high and low type agents.

Condition (5): $$\delta^{H} \geq \delta^{*} := \frac{1}{1+p(k-1)} > \delta^{L}. \tag{5}$$

Any observable deviation from the equilibrium path can be deterred by the threat of reversion to autarky. It is also easy to verify that $\underline{u}_{em} = p$ for $\delta^{H} = \delta^{*}$ so the individual rationality constraints of $\bar{u}_{em}, \underline{u}_{em} \geq p$ are satisfied. Therefore, this EM strategy profile is a Nash equilibrium. In fact, we could use the stronger equilibrium concept of *public perfect equilibrium* (PPE) following Fudenberg, Levine and Maskin.[5] A strategy for agent $i \in \{a,b\}$ is public if it depends only on her current period private information, in this case whether or not the agent received a favor opportunity, and the public history, which for this game consists of public favors done up to and including the last period. A PPE is a profile of public strategies that form a Nash equilibrium for each period and the corresponding public history.

Since the payoff pair $(\underline{u}_{em}, \bar{u}_{em})$ is enforceable (implementable), it follows by symmetry that $(\bar{u}_{em}, \underline{u}_{em})$ is also enforceable, and therefore any utility pair on the line connecting $(\underline{u}_{em}, \bar{u}_{em})$ and $(\bar{u}_{em}, \underline{u}_{em})$ is enforceable with the use of a public randomization device. AB [1] aptly call these PPE with current and continuation payoffs restricted onto a symmetric line, *symmetric self-generating line* (SSGL) equilibria. The details of SSGL and the corresponding equilibria will be explained in more detail in the next two subsections. For now it suffices to say that AB [1] solve for the highest such line; the *highest symmetric self-generating line* (HSSGL) and they show that condition (5) is exactly the right bound necessary to implement HSSGL equilibria. In fact, the simple EM mechanism is a HSSGL equilibrium for $\delta^{H} = \delta^{*}$. In the case where $\delta^H > \delta^*$, we may use the extra wiggle room to obtain a higher total payoff (and thus a higher SSGL) by requiring the advantaged agent to make further small favors while she waits for reciprocation from the disadvantaged agent.[1]

Observe that for the first-best outcome both agents would have to exhibit full trust in terms of x and y. AB [1] (p. 12) call $x + y$ the *level of trust*. Agent a (b) is said to exhibit *more trust* if $x > y$ $(y > x)$. However, if both agents exhibit full trust every period regardless of history, neither agent has any incentive to do costly favors. Thus the first-best outcome cannot be achieved. However, on the HSSGL line, the level of trust is maximized subject to the restriction that continuation payoffs are picked from the same HSSGL.

It is perhaps natural to wonder if it is incentive compatible for low types to trade smaller favors, that is, to cooperate on a lower SSGL. It is not. Decreasing the size of favors and repeating the analysis for the EM mechanism shows that discount factors above or equal to δ^* are still necessary to sustain cooperation. Furthermore, cooperation on a lower line would be less efficient. The discount factor required to support EM equilibria is independent of favor size because agents have linear utility functions. They are effectively risk neutral with respect to the size of favors. In terms of the mathematics, the lower cost of smaller favors is directly proportional by factor one to the resulting lower continuation payoffs. A formal proof will follow after we first introduce additional notation. For future reference, let $\sigma_{em(z)}$ denote the EM strategy profile when the size of exchanged favors is $z \in (0, 1]$. Let $\bar{u}_{em(z)}$ and $\underline{u}_{em(z)}$ denote respective continuation payoffs for advantaged and disadvantaged agent. Unless otherwise noted, we use EM to refer to matching of full favors, or simply full EM.

2.1 Summary of notation and structure

The notation that follows is necessary to formally define the equilibrium profiles we will use in the sections that follow, but we present it in a format well-suited for reference. Payoffs are in average discounted values.

2.1.1 Information structure:

Let $t = 1, 2, \ldots$ denote the time index. Let $w_t^i = 1$ if agent i receives a favor opportunity in period t and 0 otherwise. Agent i privately observes $W_t^i = \{w_z^i\}_{z=1}^t$. Let $\tau_t = (x, y)$ denote favors $(x, y) \in (0, 1]^2$ agents a and b, respectively, do in period t. If neither agent does a favor, then let $\tau_t = 0$. Both agents observe $T_t = \{\tau_z\}_{z=1}^t$. Private history of agent i and public history up to and including period t are denoted by $h_t^i = W_t^i \in \mathcal{H}_t^i$ and $H_t = T_t \in \mathcal{H}_t$, respectively. A strategy for agent i, denoted by σ^i, consists of a favor making decision, I_t^i, for each period based on i's type, her private history up to period t, and public history up to period $t - 1$. More formally, $I_t^i : \{H, L\} \times \mathcal{H}_t^i \times \mathcal{H}_{t-1} \to [0, 1]$ s.t. $I_t^i = 0$ when $w_t^i = 0$.

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[1] AB also constructed other types of equilibria that may lead to higher or lower total payoffs than HSSGL equilibria depending on the parameter values. However, we concentrate on HSSGL equilibria because they always exist if condition (5) is satisfied, and loosely speaking outperform other types of equilibria when p is not very high.

Model parameters:	
$i \in \{a, b\}$:	Agents.
$\omega^i \in \{L, H\}$:	Agent i's type; L = low, H = high.
$p \in (0, 1)$:	Probability agent $i \in \{a, b\}$ receives a favor opportunity. Favor opportunities are either mutually exclusive or independent.
$k > 1$:	Benefit per unit of favor.
$\delta^i \in (0, 1)$:	Discount factor of agent $i \in \{a, b\}$.
$\mu_o \in (0, 1)$:	Fraction of high type agents in population.
Actions:	
$x, y \in [0, 1]$:	Size of favor by agents a, b, respectively.
Payoffs:	
(u, v) :	Current payoffs to agents (a, b).
(u_o, v_o) :	Continuation payoffs to (a, b) when no one does a favor.
(u_i, v_i) :	Continuation payoffs to (a, b) when $i \in \{a, b\}$ does a favor.

Table 1: Summary of notation

2.2 Strategies and equilibrium concepts

For our solution concept we will use *Perfect Bayesian equilibrium* (PBE). PBE consist of a strategy profile $\left(\sigma = \left(\sigma^a, \sigma^b\right)\right)$ and a belief system $\left(\mu = \left(\mu^a, \mu^b\right)\right)$ such that σ is sequentially rational with respect to μ and μ is consistent with σ. That is, the strategies are optimal at every stage of the game given the beliefs, and the beliefs are updated according to Bayes' rule from equilibrium strategies and observed actions. We should, strictly speaking, also specify beliefs for off-equilibrium path actions, however, we deter these actions with the threat of autarky play, which is always an equilibrium response, so it is understood that beliefs consistent with autarky exist and would be straightforward if burdensome to specify. Therefore we generally leave out off-equilibrium path beliefs from our belief functions. But this brings us to the following two definitions.

Definition 1 *Let σ^i_{aut} be such that $I^i_t = 0, \forall t$.*

Definition 2 *Let $\mathcal{H}^*_t$ be the set of all public on-equilibrium path histories up to and including period t.*

For example, if two agents are playing a full equality matching game and agent a is the initial disadvantaged agent, any history such that agent b did the first favor, one of the agents did two consecutive favors, or a partial favor, would not be in $\mathcal{H}^*_t$. However, histories that include only private deviations, that is, a disadvantaged agent does not do a favor when she has the opportunity, would still be in $\mathcal{H}^*_t$. Next, let us define EM formally.

Definition 3 *An equality matching strategy at level $z \in (0, 1]$ for agent i, denoted by $\sigma^i_{em(z)}$ or simply σ^i_{em} when $z = 1$, is such that*

$$I^i_t = \begin{cases} z & \textit{if agent } i \textit{ is disadvantaged, } w^i_t = 1 \textit{ and } h_{t-1} \in \mathcal{H}^*_{t-1}, \\ 0 & \textit{otherwise.} \end{cases}$$

Lemma 4 (Necessary and sufficient condition for EM) *$\delta^H \geq \delta^* = \frac{1}{1+p(k-1)}$ is necessary and sufficient to implement $\sigma_{em(z)}, \forall z \in (0,1]$ in a complete information environment.*

Proof. In appendix. ■

While EM is generally not the most efficient way to trade favors, the first best outcome is not enforceable, and in the AB [1] model, the second best outcome may be intractable. Presumably that is why AB [1] focused on PPE restricted to symmetric lines rather than to the whole space of feasible and individually rational payoffs. While our primary interest is to implement separation efficiently, rather than to optimize subsequent endgames, we do argue that after separation into an EM game, high types can achieve equilibria of higher value. To this end, we explain AB's *highest symmetric self-generating line* (HSSGL) equilibrium concept [1] as it applies to our version of their model. While we do not repeat their proofs, we do provide a basic explanation of how these results were obtained because our model is sufficiently different that a direct transition of results from AB [1] would not be immediate or even possible. Later we show formally that a pair of high types can move to a HSSGL equilibrium after equality matching for long enough.

Definition 5 *A self-generating line (SGL) is a line in the payoff space such that any payoffs (u,v) on the line may be implemented using some actions (x,y) and continuation payoffs (u_i, v_i, u_o, v_o) subject to $u_i + v_i = T$ and $u_o + v_o = T$ for $i \in \{a,b\}$. A symmetric self-generating line is a SGL such that $(\underline{u}, \bar{u}) \rightarrow (\bar{u}, \underline{u})$, and the highest symmetric self-generating line (HSSGL) is a symmetric SGL such that T is maximized [1] (p. 12).*

The HSSGL equilibria are PPE restricted to symmetric lines. As is normal in the literature, AB [1] use the recursive approach introduced by Abreu, Pearce and Stacchetti [2] (APS). Let operator B produce the largest self-generating set of PPE values, Ψ^*.[2] Then for any set $\Psi \subset \mathbb{R}^2$, let $B(\Psi) = \{(u,v) \in \Psi : \exists (u_i, v_i) \in \Psi, i \in \{a,b\}; (u_o, v_o) \in \Psi; x, y \in [0,1]$, s.t. (6)-(10) below are satisfied$\}$. [1] (p. 10-11). In autarky each agent's payoff is p, so the individual rationality constraints are the following:

$$IR : u, v, u_i, v_i, u_o, v_o \geq p. \tag{6}$$

The relevant incentive compatibility constraints for agents a and b state that when the agent has an opportunity to do a favor, the flow payoff, which reflects the cost of the favor, and the continuation promise for doing the favor exceed the flow payoff without cost and the continuation payoff when neither agent does a favor:

$$ICC_x^a \; : \; \left(1-\delta^H\right)(1-x) + \delta^H u_a \geq \left(1-\delta^H\right) + \delta^H u_o, \tag{7}$$
$$ICC_y^b \; : \; \left(1-\delta^H\right)(1-y) + \delta^H v_b \geq \left(1-\delta^H\right) + \delta^H v_o. \tag{8}$$

The current period payoffs u and v must be consistent with the flow payoffs and the continuation promises consisting of three possible outcomes: either agent i receives a favor opportunity

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[2] To this end, assume that observable off-equilibrium path deviations, such as a favor of different size from the on-equilibrium path favor, will be punished with reversion to autarky. Thus, the only relevant deviations for the following analysis will be on-equilibrium path deviations, namely deviations that are not observable to one of the agents.

and does a favor for j, agent j receives a favor opportunity and does a favor for i, or neither agent receives a favor opportunity:

$$u = p\left(\left(1-\delta^H\right)(1-x)+\delta^H u_a\right)+p\left(\left(1-\delta^H\right)ky+\delta^H u_b\right)+(1-2p)\,\delta^H u_o, \tag{9}$$
$$v = p\left(\left(1-\delta^H\right)(1-y)+\delta^H v_b\right)+p\left(\left(1-\delta^H\right)kx+\delta^H v_a\right)+(1-2p)\,\delta^H v_o. \tag{10}$$

By restricting their analysis onto symmetric lines, AB [1] essentially reduce the problem to starting with the full EM profile, σ_{em}, and solving for the highest enforceable favor by the advantaged agent, that is a favor that makes her incentive compatibility problem bind, and all continuation payoffs must be chosen from the same symmetric line. That way the value of the game remains the same regardless of who, if anyone, does a favor. In particular, AB [1] (p. 24-25) provide the equivalent of the following characterization of their HSSGL equilibrium, which implies that the corner payoff pair, $(\underline{u},\bar{u})$, can be implemented with $(x,y,u_a,u_b,u_o,v_a,v_b,v_o)$, such that

$$x = 1,\ y=\tfrac{\delta^H-\delta^*}{\delta^H+\delta^*}, \tag{11}$$
$$u_a = \underline{u}+(1+y)\tfrac{1-\delta^H}{\delta^H}=\bar{u},\ u_b=\underline{u},\ u_o=\underline{u}+y\tfrac{1-\delta^H}{\delta^H}, \tag{12}$$
$$v_a = \underline{u},\ v_b=\bar{u},\ v_o=\bar{u}-y\tfrac{1-\delta^H}{\delta^H}, \tag{13}$$
$$\underline{u} = p+y\left(p\left(k-1\right)+1\right)=p+\tfrac{\delta^H-\delta^*}{\delta^H+\delta^*}\tfrac{1}{\delta^*}, \tag{14}$$
$$\bar{u} = pk-y=\underline{u}+\tfrac{2\left(1-\delta^H\right)}{\delta^H+\delta^*}. \tag{15}$$

and by the threat of autarky if one of the agents publicly deviates off-equilibrium path. The other corner payoff pair, $(\bar{u},\underline{u})$, can be implemented symmetrically. It then follows that any payoff pair (u,v) between $(\underline{u},\bar{u})$ and $(\bar{u},\underline{u})$ may be implemented with a public randomization device at the start of the game. Call this strategy profile $\sigma_{hssgl}(u,v)$.

In a 2005 version of their paper, AB [1] further calculate a deterministic algorithm to enforce $\sigma_{hssgl}(u,v)$. From the corner solution we know the total trust, $x+y=\frac{2\delta^H}{\delta^H+\delta^*}$, and the total payoff, $T=u+v=2p+\frac{2\delta^H p(k-1)}{\delta^H+\delta^*}$, $\forall x,y,u,v$ on the HSSGL. We also have the corner point values, $(\underline{u},\bar{u})$ and $(\bar{u},\underline{u})$, defined by (14) and (15), that are necessary for use as maximal rewards to implement any HSSGL equilibria. In fact, we know that $u_a=\underline{u}+(x+y)\frac{1-\delta^H}{\delta^H}=\bar{u}, u_b=\underline{u}, v_a=\underline{u}$ and $v_b=\bar{u}$. The incentive compatibility constraints must also bind, so we can use identities mentioned previously, along with (binding) equations (7)-(10) to calculate x,y,u_o and v_o required to implement any payoff pair (u,v) on the HSSGL defined by $(\underline{u},\bar{u})$ and $(\bar{u},\underline{u})$. Our equivalents of AB's equations [1] are stated below without proof:[3]

$$x = \tfrac{\delta^H\delta^*}{1-\delta^H}\left[\tfrac{v-\left(1-\delta^H\right)p}{\delta^H}-\underline{u}\right],\ y=\tfrac{\delta^H\delta^*}{1-\delta^H}\left[\tfrac{u-\left(1-\delta^H\right)p}{\delta^H}-\underline{u}\right], \tag{16}$$

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[3] A strategy profile $\sigma(u,v)$ that begins at time $t\in\mathbb{N}$, denotes a strategy profile σ that implements expected payoffs (u,v) at that time. However, σ will not implement (u,v) every period after t. The payoffs to be implemented each period depend on the history and σ. For example, $\sigma_{hssgl}\left(\underline{u},\bar{u}\right)$ at $t=1$, denotes a HSSGL strategy profile that at the beginning of period 1 is expected to generate payoffs $(\underline{u},\bar{u})$. If neither agent does a favor in the first period, σ_{hssgl} is expected to yield payoffs $\left(u_o\left(\underline{u},\bar{u}\right),v_o\left(\underline{u},\bar{u}\right)\right)$ at the beginning of period 2, where u_o and v_o are given by the last equation above for $u=\underline{u}$ and $v=\bar{u}$.

$$u_o = \delta^* \left[\frac{u-\left(1-\delta^H\right)p}{\delta^H} + \frac{\underline{u}(1-\delta^*)}{\delta^*} \right], \; v_o = \delta^* \left[\frac{v-\left(1-\delta^H\right)p}{\delta^H} + \frac{\underline{u}(1-\delta^*)}{\delta^*} \right]. \quad (17)$$

2.3 Previous favor-trading literature

Möbius [12] first investigated the type of 2-player favor-trading games we study (2001), albeit with complete information and continuous time. He focused on an intuitive "chips mechanism." That is, each player begins with K chips, and each time an agent does a favor, she earns a chip. If one agent accumulates all $2K$ chips, she suspends favors until reciprocation. EM is effectively a chips game with only one chip held by the advantaged agent. Hauser and Hopenhayn [8] continue Möbius' [12] favor-trading research by allowing partial favors (2005). Consequently, they let the cost of favors vary based on public history of favor exchanges, notably including time passed since the last exchange. They characterize a set of Pareto optimal PPE, and show numerically that partial favors lead to significant efficiency gains over Möbius' chips mechanism.[12] Their findings display similar characteristics to HSSGL equilibria formulated by AB [1] in discrete time (first draft in 2004). Both Hauser and Hopenhayn [8] and AB [1] use PPE as their solution concept and allow partial favors. Both find equilibria that call for larger favors to be followed by unlimited smaller favors until reciprocation. This is in contrast to Möbius [12] who assumed favors were all the same size, and an agent would suspend favors whenever she was owed $2K$ favors. Both Hauser and Hopenhayn [8] and AB [1] discover what the former call "debt forgiveness." That is, the value of favors owed declines; debt is forgiven, unless "interest" in the form of small favors is "paid" by the advantaged agent.

Therefore it seems that favor-trading in a complete information environment is robust to the model's timing structure (continuous versus discrete) and the arrival process of favors (independent versus mutually exclusive). We show that with incomplete information, the more efficient equilibria characterized by multiple consecutive favors are initially precluded by the possibility of low types, but can always be achieved over time between high types. We also show that the distribution of the arrival process of favors plays an important role in strategies for separation. Furthermore, independent favor opportunities in discrete time raise the possibility of both agents having the opportunity to do a favor for at the same time. This possibility has not been studied in previous favor-trading models.

A notable difference between Hauser and Hopenhayn [8] and AB [1] is that AB include opportunities for immediate reciprocity with private information. However, AB [1] show that immediate reciprocity is unnecessary for HSSGL equilibria, which is why our streamlined version of their model does not include it. Further, AB [1] describe favor opportunities as income shocks, and favors as investments. We dropped this terminology because favor-trading precludes side payments, and we felt that using monetary language to discuss the topic confused the issue. We also normalized payoffs to average discounted values for convenience.

Outside of the favor-trading literature, Watson [19] did study the sustainability of cooperation using a two-player infinitely-repeated prisoner's dilemma model with incomplete information about agents' types. However, in his model deviations from cooperative behavior are publicly observable, whereas in favor-trading games only cooperative actions are observable, and deviations are private Still, both models have the broad characteristic that agents start cooperation with small stakes, but form more profitable relationships over time if each agent proves her willingness to cooperate.

3 One-sided incomplete information (mutually exclusive favor opportunities)

For the rest of this section, we consider the case of mutually exclusive favor opportunities, with one known agent; a is a high type, and one unknown agent; b is a high type with probability μ_o, and a low type with probability $1-\mu_o$.

3.1 Separation with equality matching (EM) equilibria

A low type agent b facing a high type agent a would prefer to be seen as a high type in order to receive favors from a even though she would not reciprocate. The high type wants to separate herself from the low type as soon as possible and exchange favors with a. The question is whether a separating equilibrium exists, and if so, how quickly and efficiently can a high type separate?

The answer to the first part should be clear if we consider the EM strategy profile, $\sigma_{em(z)}\left(\bar{u}_{em(z)},\underline{u}_{em(z)}\right)$, discussed in section 2. Recall that the necessary and sufficient condition for agent b to benefit from EM is $\delta^b \geq \delta^*$. It follows immediately that a high type may separate as soon as she receives a favor opportunity by doing a favor of size z. Call this *immediate separation,* that is, separation at first possible opportunity. To formalize this answer, we first have to define beliefs.

Definition 6 $\mu \equiv \mu_t \equiv \left(\mu_t^a,\mu_t^b\right)$ *where* $\mu_0^i \equiv \mu_o$ *and* $\mu_t^i : \mathcal{H}_t^i \times \mathcal{H}_{t-1} \to [0,1]$ *represents agent* i*'s belief. That is,* μ_t^i *is the probability assigned by* i *to the event that the other agent is a high type based on* i*'s private history up to period* t *and public history up to* $t-1$.

Sometimes we drop the time index for convenience $(\mu \equiv \mu_t)$. The domain of the belief function consists of agent i's private history upto the current period and the public history upto the last period because it refers to agent i's belief at a point in period t when i has observed her private signal (her favor opportunity is 0 or 1) but not the public signal (period t favors, if any, are still pending). That is, μ_t^i captures agent i's updated belief in period t at the point in time when she has either received a favor opportunity and is deciding whether or not to do a favor, or she has received no opportunity and is waiting to see if the other agent does a favor.

Lemma 7 (Separation with one-sided incomplete information) *A strategy profile* (σ,μ) *defined by equations (18)-(21) is a PBE for* $z \in (0,1]$.

$$\sigma^a := \sigma^a_{em(z)}\left(\bar{u}_{em(z)},\underline{u}_{em(z)}\right), \tag{18}$$

$$\sigma^b := \begin{cases} \sigma^b_{em(z)}\left(\bar{u}_{em(z)},\underline{u}_{em(z)}\right) & \text{if } \omega^b = H, \\ \sigma^b_{aut} & \text{if } \omega^b = L, \end{cases} \tag{19}$$

$$\mu_t^b := 1, \tag{20}$$

$$\mu_t^a := \begin{cases} 0 & \text{if } H_{t-1} \notin \mathcal{H}^*_{t-1} \text{ (off-equilibrium path move)}, \\ 1 & \text{else if } \exists n < t \text{ s.t. } \tau_n = (0,z), \\ \mu^a_{t-1} & \text{else if } w_t^a = 1, \\ \frac{\mu^a_{t-1}(1-2p)}{1-\left(1+\mu^a_{t-1}\right)p} & \text{otherwise.} \end{cases} \tag{21}$$

Proof. The proof follows immediately from lemma 4 as long as σ is consistent with μ. In this case μ was constructed from σ by Bayesian updating, so consistency follows. Recall that part of the EM strategy is to stop doing favors (switch to autarky) if anyone deviates publicly from the equilibrium path. Thereby such moves are deterred. Consequently off-equilibrium path beliefs are moot (first row of μ_t^a). On the equilibrium path, three possibilities exist: (i) agent b does a favor of size z, in which case a believes b is a high type (second row of μ_t^a). This belief is consistent with σ because only a high type would do a favor per σ. (ii) If agent a receives the favor opportunity, b does not because favor opportunities are mutually exclusive, so a's belief about b does not change (third row of μ_t^a). This is trivially consistent with σ. (iii) Agent a receives neither a favor opportunity, nor a favor from b. Either agent b did not receive a favor opportunity (neither agent did), or she received the opportunity but did not do the favor per σ because she is a low type. In this case, agent a's updated belief per Bayes' rule is

$$\begin{aligned}\mu_t^a &= P\left(\omega^b = H : \{\tau_t = 0 \cap w_t^a = 0\}\right) = \tfrac{P\left(\omega^b=H \cap \tau_t=0 \cap w_t^a=0\right)}{P(\tau_t=0 \cap w_t^a=0)} \\ &= \tfrac{\mu_{t-1}^a(1-2p)}{\mu_{t-1}^a(1-2p)+\left(1-\mu_{t-1}^a\right)(p+1-2p)} = \tfrac{\mu_{t-1}^a(1-2p)}{1-\left(1+\mu_{t-1}^a\right)p}.\end{aligned}$$

The probability tree below depicts paths to potential events that agent a may observe and her subjective equilibrium beliefs along these paths. The dotted lines connect outcomes that are observationally equivalent for agent a. ■

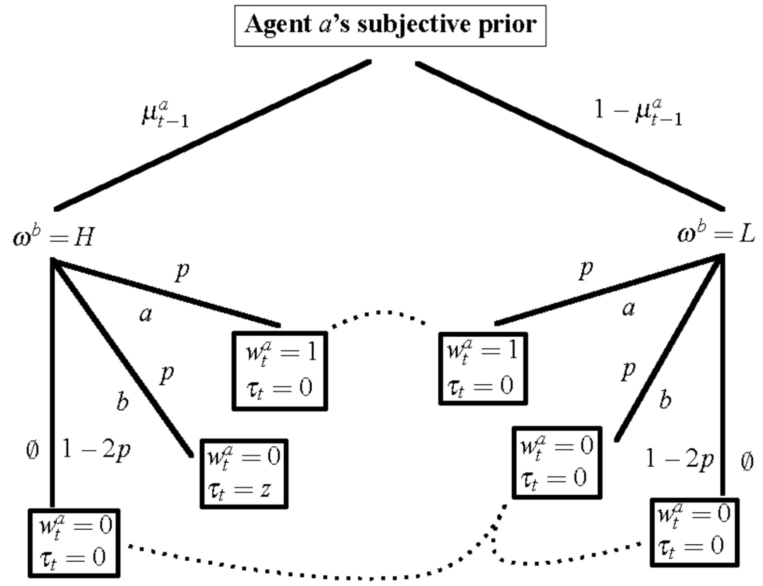

Figure 1: Stage game event tree over subjective beliefs

3.2 Separation and highest symmetric self-generating line (HSSGL) equilibria

Next we investigate whether or not high types can separate immediately into a HSSGL equilibrium profile. If $\delta^H = \delta^*$, full EM and HSSGL equilibria coincide. This was shown in section

2. Therefore, suppose $\delta^H > \delta^*$ for the rest of this subsection. As before we assume agent a is a high type and agent b is an unknown type.

For the moment, suppose agent a's strategy is to wait for a favor of size $z \in (0,1]$, and if b does such a favor, a will respond with HSSGL play with b as the advantaged player. The point of the exercise is to find out if a HSSGL equilibrium can be directly implemented after separation. To this end, we may set $z = 1$ without compromising the generality of the result. This choice of z minimizes the incentive for low types to mimic high types, yet is incentive compatible for high types per subsection 2.2, in which AB's HSSGL equilibrium [1] results were adapted to our model with complete information. In particular, equations (12)-(15) detail the incentive compatible HSSGL strategy profile to support a corner solution. That is, the continuation payoffs necessary to support a full favor by a disadvantaged high type agent, and a partial favor by the advantaged agent. Recall that each high type agent's incentive compatibility constraint binds in a HSSGL equilibrium, or else we could use the available slack to move to a higher line. Therefore, we cannot lower the continuation promise to agent b to deter low types further unless we deter the high types as well.

In summary, it is enough to calculate if a low type agent b would be willing to do a full favor in return for $\sigma^a_{hssgl}(\underline{u}, \bar{u})$. Equations (16) and (17) specify all actions by agent a, and b follows the autarky strategy after the initial full favor. Therefore we may calculate the expected payoff to agent b and verify that for some parameter values she would have incentive to pool with high types. The lemmas below formalize these claims.

Lemma 8 (HSSGL payoff to advantaged low types) *The expected payoff to a low type agent* b *under strategy profile* $\sigma = \left(\sigma^a_{hssgl}(\underline{u}, \bar{u}), \sigma^b_{aut}\right)$ *is*

$$\overline{v}^L_0 = p + pk\frac{\delta^H\left(\delta^H+\delta^*-2\delta^L\delta^*\right)}{\left(\delta^H+\delta^*\right)\left(\delta^H-\delta^L\delta^*(1-p)\right)}. \tag{22}$$

Proof. In appendix. ■

The next proposition defines a strategy for agent a of doing no favors unless b does a full favor first. If the favor is received, a will play according to HSSGL strategy profile that implements the corner payoff pair $(\underline{u}, \bar{u})$ favoring agent b. Given such a strategy, the proposition will prove that condition (5) alone does not guarantee that high types can separate immediately into a HSSGL endgame, and a stronger condition is required.

Proposition 9 (Immediate HSSGL separation) *Consider HSSGL payoffs* $(\underline{u}, \bar{u})$ *defined by (14)-(15) and let* t^* *denote the first time agent* b *does a full favor. Then for strategy*

$$\sigma^a := \begin{cases} \sigma^a_{aut} & \text{for } t \leq t^* \text{ (no favors until } b \text{ does a full favor)} \\ \sigma^a_{hssgl}(\underline{u}, \bar{u}) & \text{starting at } t = t^*+1 \text{ (HSSGL play if } b \text{ does a favor)} \end{cases}$$
$$\text{where } t^* := \inf\{t \in \mathbb{N} : \tau_t = (0,1)\} \text{ and } \inf\{\emptyset\} \equiv \infty,$$

immediate separation is enforceable only if δ^L *is low enough or* δ^H *is high enough. The technical condition they must is*

$$\frac{\delta^H k\ p(1+(k-1)p)\left(1-2\delta^L+\delta^H(1+(k-1)p)\right)}{\left(1+\delta^H(1+(k-1)p)\right)\left(\delta^H(1+(k-1)p)-\delta^L(1-p)\right)} \geq \frac{1-\delta^L}{\delta^L}. \tag{23}$$

Proof. Given σ^a a low type agent b will only do a full favor if the following incentive compatibility constraint is satisfied:

$$ICC^L_{hssgl}: \qquad \delta^L \overline{v}^L_0 \geq 1 - \delta^L + \delta^L p. \tag{24}$$

Substituting in for $\overline{v}^L_0$ from lemma 8 and simplifying yields condition (23). ■

Lemma 10 (Simple lower bound for $\bar{\delta}^L$) *Let $\bar{\delta}^L$ be defined by (25),*

$$\delta^L \leq \bar{\delta}^L := \frac{2}{\hat{B}+\sqrt{\hat{B}^2-\frac{4\alpha\left(\hat{B}+p(k-1)\right)}{1+\alpha}}} < \delta^* \tag{25}$$
$$\text{where } \hat{B}=1+\alpha(1-p)+pk \text{ and } \alpha\equiv\delta^*/\delta^H.$$

then $\bar{\delta}^L \geq \underline{\delta}^L := 1/\left(1+pk\right).$

Proof. The expressions for $\bar{\delta}^L$ is derived in the appendix. The continuation promise for a low type that mimics a high type is in HSSGL is $\overline{v}^L_0 = p\left(k+1\right)$ because with probability p she receives a favor of value k from the other agent and with probability p she receives the favor opportunity of value 1. Substituting $\overline{v}^L_0 = p\left(k+1\right)$ into ICC^L_{hssgl} (inequality 24), assuming it binds, and solving for δ^L produces solution $\delta^L = 1/\left(1+pk\right)$. Therefore it suffices to show that $\overline{v}^L_0 \leq p\left(k+1\right)$. Rewriting $\overline{v}^L_0$ from lemma 8:

$$\begin{aligned}
\overline{v}^L_0 &= p + pk\frac{\delta^H\left(\delta^H+\delta^*\right)-2\delta^H\delta^L\delta^*}{\delta^H\left(\delta^H+\delta^*\right)-(1-p)\left(\delta^H+\delta^*\right)\delta^L\delta^*} \\
&\underset{\because 2\delta^H\delta^L\delta^*>(1-p)\left(\delta^H+\delta^*\right)\delta^L\delta^* \Longleftarrow 2>(1-p)\left(1+\delta^*/\delta^H\right)}{<} p + pk\frac{\delta^H\left(\delta^H+\delta^*\right)-(1-p)\left(\delta^H+\delta^*\right)\delta^L\delta^*}{\delta^H\left(\delta^H+\delta^*\right)-(1-p)\left(\delta^H+\delta^*\right)\delta^L\delta^*} \\
\Longrightarrow \overline{v}^L_0 &< p\left(k+1\right).
\end{aligned} \tag{26}$$

$$\Longrightarrow \bar{\delta}^L \in \left(\tfrac{1}{1+pk}, \tfrac{1}{1+p(k-1)}\right) \equiv \left(\underline{\delta}^L, \delta^*\right). \blacksquare$$

The point of presenting lemma 10 was to demonstrate that for k large, $\underline{\delta}^L \rightarrow \delta^*$ and therefore $\bar{\delta}^L \rightarrow \delta^*$. Suppose that $\delta^L \sim \mathcal{U}\left(0,\delta^*\right)$, then the subpopulation of low types with incentive to mimic high types in a game involving separation into HSSGL is bounded from above by $\delta^* - \underline{\delta}^L$ per lemma 10. As a fraction of all low types

$$\begin{aligned}
\tfrac{\delta^*-\underline{\delta}^L}{\delta^*} &= \frac{\frac{1}{1+p(k-1)}-\frac{1}{1+pk}}{\frac{1}{1+p(k-1)}} = \frac{\frac{p}{(1+pk)(1+p(k-1))}}{\frac{1}{1+p(k-1)}} \\
&= \tfrac{p}{(1+pk)} \rightarrow 0 \text{ as } k \rightarrow \infty.
\end{aligned}$$

The discount factor bounds, δ^* and $\bar{\delta}^L$, are directly proportional to the expected continuation payoff from $\left(\sigma^a_{em}\left(\underline{u}_{em}, \bar{u}_{em}\right), \sigma^b_{aut}\right)$ and $\left(\sigma^a_{em}\left(\underline{u}, \bar{u}\right), \sigma^b_{aut}\right)$, respectively. Since the difference between δ^* and $\bar{\delta}^L$ is generally speaking small, it follows that the low type would generally not stand to benefit greatly from HSSGL play over EM play. The reason is that in EM the advantaged low type is guaranteed a favor worth k as soon as agent a receives a favor opportunity, whereas in HSSGL play, low type agent b will only receive a full favor if agent a receives a favor opportunity the very next period after b gains the advantaged status. After that the size of the favor b is owned keeps decreasing unless b keeps doing small favors to remain fully advantaged, and even then her continuation payoff would depreciate each period when neither agent received a favor opportunity.

As for condition (23), we can write $\tilde{\delta}^H = \frac{k}{(1+p(k-1))^2} - \frac{1}{1+p(k-1)}$. It is then easy to see that $\tilde{\delta}^H \to 0$ as $k \to \infty$. The exception to this rule is if $p \to 0$ at a faster rate than $1/\sqrt{k}$. That is, condition (23) is generally violated only if p is significantly smaller than k in relative terms. The reason why δ^H is even a factor in the low types incentive compatibility constraints is because of the way HSSGL equilibria are calculated; namely, the size of partial favors are chosen so that ICC of high types binds. The higher the high type's discount factor, the greater the favors. We have now established conditions under which low types would not mimic high types. Next we use these results to formally define an equilibrium profile that implements immediate separation if agent b is a high type and $\delta^L \leq \bar{\delta}^L$.

Lemma 11 (From EM to HSSGL) *Suppose $\delta^L \leq \bar{\delta}^L$ as defined by equations (25). Let $t^* := \inf\{t \in \mathbb{N} : \tau_t = (0,1)\}$ where $\inf\{\emptyset\} \equiv \infty$, and consider strategy profile (σ, μ) defined by equations (27)-(30).*

$$\sigma^a := \begin{cases} \sigma^a_{aut} & \textit{for } t \leq t^*, \\ \sigma^a_{hssgl}\left(\underline{u}, \bar{u}\right) & \textit{for } t = t^* + 1, \end{cases} \tag{27}$$

$$\sigma^b := \begin{cases} \sigma^b_{hssgl}\left(\bar{u}, \underline{u}\right) & \textit{starting from } t = \inf\left\{s \in \mathbb{N} : w^b_s = 1\right\} \textit{ if } \omega^b = H, \\ \sigma^b_{aut} & \textit{otherwise,} \end{cases} \tag{28}$$

$$\mu^b_t := 1, \tag{29}$$

$$\mu^a_t := \begin{cases} 0 & \textit{if } H_{t-1} \notin \mathcal{H}^*_{t-1} \textit{ (off-equilibrium path move),} \\ 1 & \textit{if } H_{t-1} \in \mathcal{H}^*_{t-1} \textit{ and } \exists n < t \textit{ s.t. } \tau_n = (0, z), \\ \mu^a_{t-1} & \textit{if } w^a_t = 1, \\ \frac{\mu^a_{t-1}(1-2p)}{1-\left(1+\mu^a_{t-1}\right)p} & \textit{otherwise.} \end{cases} \tag{30}$$

Then (σ, μ) is a PBE profile.

Proof. Since $\sigma_{hssgl}\left(\underline{u}, \bar{u}\right)$ is a PPE profile (restricted to a symmetric line) with complete information, it is also a PBE profile post separation since beliefs become trivial at that point, and $\sigma_{hssgl}\left(\underline{u}, \bar{u}\right)$ implements a Nash equilibrium at each stage of the game that follows. For $t \geq t^*$, $\mu^a = \mu^b = 1$ on the equilibrium path, so the beliefs are consistent with $\sigma_{hssgl}\left(\underline{u}, \bar{u}\right)$. And given σ^a, it is incentive compatible for a high type agent b to do a full favor $(y = 1)$ when she first receives a favor opportunity per subsection 2.2, and equations (11)-(15) in particular. The belief system μ is clearly consistent with σ, and given $\bar{\delta}^L$, low types would not try to mimic the high types per proposition 9. ■

The next step is to consider what happens if low types have discount factors between $\bar{\delta}^L$ and δ^*. We know immediate separation can always be implemented with an EM endgame per lemma 7, but not with a HSSGL endgame per proposition 9. However, since δ^* is the threshold discount factor required for equality matching to be incentive compatible, and $\delta^L < \delta^*$ by definition, it follows that EM is strictly not incentive compatible for low types. Suppose we promise an agent a payoff T periods from now preceded by full EM. This strategy profile would be incentive compatible for high types for any T in the absence of low types, so we can make T sufficiently large that low types are deterred by the prospect of T periods of EM play prior to the promised higher value payoff.

Corollary 12 *Suppose* $\delta^L \in (\bar{\delta}^L, \delta^*)$. *Let* $t^* := \inf\{t \in \mathbb{N} : \tau_t = (0,1)\}$, *where* $\inf\{\emptyset\} \equiv \infty$, *and consider* (σ, μ) *defined by (31)-(34).*

$$\sigma^a = \begin{cases} \sigma^a_{em}\left(\bar{u}_{em}, \underline{u}_{em}\right) & \textit{for } t < t^* + T, \\ \sigma^a_{hssgl}\left(\bar{u}, \underline{u}\right) & \textit{starting at } t = t^* + T \textit{ if } a \textit{ advantaged,} \\ \sigma^a_{hssgl}\left(\underline{u}, \bar{u}\right) & \textit{starting at } t = t^* + T \textit{ if } b \textit{ advantaged,} \end{cases} \tag{31}$$

$$\sigma^b = \begin{cases} \sigma^b_{em}\left(\bar{u}_{em}, \underline{u}_{em}\right) & \textit{if } \omega^b = H \textit{ starting at } t = \inf\left\{s \in \mathbb{N} : w^b_s = 1\right\} \\ & \quad \textit{for } T \textit{ periods,} \\ \sigma^b_{hssgl}\left(\bar{u}, \underline{u}\right) & \textit{starting at } t = t^* + T \textit{ if } a \textit{ advantaged and } \omega^b = H, \\ \sigma^b_{hssgl}\left(\underline{u}, \bar{u}\right) & \textit{starting at } t = t^* + T \textit{ if } b \textit{ advantaged and } \omega^b = H, \\ \sigma^b_{aut} & \textit{if } \omega^b = L, \end{cases} \tag{32}$$

$$\mu^a_t = \begin{cases} 0 & \textit{if } h_{t-1} \notin H^*_{t-1}, \\ 1 & \textit{if } t \geq t^* \textit{ and } h_{t-1} \in H^*_{t-1}, \\ \mu^a_{t-1} & \textit{if } w^a_t = 1, \\ \frac{\mu^a_{t-1}(1-2p)}{1-\left(1+\mu^a_{t-1}\right)p} & \textit{otherwise,} \end{cases} \tag{33}$$

$$\mu^b_t = 1. \tag{34}$$

Then (σ, μ) *is a PBE for* $T \in \mathbb{N}$ *large enough.*

Proof. In appendix. ■

There are several other methods to deter low types before a move to a HSSGL endgame Some of these methods are probably more efficient in some circumstances. For example, if $\delta^H > \delta^*$, there exists some slack in the high type agent b's EM incentive compatibility constraints, that could be used to implement a lower first return favor by agent a. Given a large enough difference between δ^L and δ^H, this tactic might deter the low types after just one round of favor trades, albeit at b's expense. Another method would be to play according to a HSSGL equilibrium profile for some $\hat{\delta} \in \left(\delta^*, \delta^H\right)$. In particular, if the optimal time required to deter a low type were measured in continuous time, then it would land on a $T \in \mathbb{N}$ with probability 0. That is, a small inefficiency is generated by the discrete time structure of our model, which necessitates rounding off to the nearest integer. Such inefficiencies could be removed, at least partly, by playing according to a HSSGL equilibrium profile, in the last period of T, for an appropriately chosen $\hat{\delta} \in \left(\delta^*, \delta^H\right)$. However, while we recognize that such small efficiency gains are possible, we chose to focus on the EM separation because it is more general.

4 Two-sided incomplete information (mutually exclusive favor opportunities)

When incomplete information is two-sided, a high type agent may attempt to separate herself from the low type by doing the first favor. Denote such a favor by z_t at period t. If the initial favor necessary to trigger cooperation between high types is sufficiently large, it will deter low types from mimicking high types, providing a way for high types to separate themselves. After an initial separation, we are back to the one-sided incomplete information case described by the results in the previous section. The question is, can we find a sequence of sufficiently large initial favors to deter the low types from pooling with high types, but still low enough that high types would have incentive to do the favor and risk not receiving reciprocation if the other is a low type, instead of just waiting for the other agent to go first. Recall that in AB's EM and HSSGL equilibria [1], advantaged and disadvantaged designations are determined exogenously, so there is no first mover problem.

In the first subsection, we will follow the same approach. We investigate equilibria in which one agent is designated to do the first favor. Such a strategy takes care of the first mover problem. The designated agent, or the *designee* from now on, will not grow increasingly pessimistic when the other agent does no favors (since the strategy profile doesn't call for it), and she has no option to wait for the other to go first, so her incentives to separate are increased. However, the trade-off is that separation between high types takes roughly twice as long as it would if both agents were attempting to separate. This is why we investigate symmetric strategies for separation in the second subsection.

4.1 Designated first favor maker (DFFM) equilibria

Consider a strategy of designating an agent at the beginning of the game to do the first favor of pre-agreed size z, while the other waits. In separating equilibria z has to be high enough to deter low types from pooling with high types. If the designee does a favor of size z, she earns the advantaged status in an full EM game that follows, provided of course the other agent is a high type. A formal description of this strategy, and a proof are below. For now, suppose without loss of generality that agent a is the designee. For now we abstract from how the agents determine the designation, but for some later results we will assume agents can randomize, at least with even odds (coin flip), to see who is to do the first favor.

Proposition 13 (Existence of DFFM equilibria) *Let* $t^* := \inf\{t \in \mathbb{N} : \tau_t = (z, 0)\}$ *and consider strategy profile* (σ, μ) *be such that*

$$\sigma^a := \begin{cases} \sigma^a_{em(z)}\left(\underline{u}_{em(z)}, \bar{u}_{em(z)}\right) & \textit{if } h_{t-1} \in H^*_{t-1}, \omega^a = H \textit{ and} \\ & \quad t \leq \inf\{s \in \mathbb{N} : w^a_s = 1\}, \\ \sigma^a_{em}\left(\bar{u}_{em}, \underline{u}_{em}\right) & \textit{starting at } t = t^* \textit{ if} \\ & \quad h_{t-1} \in H^*_{t-1}, \omega^a = H, \\ \sigma^a_{aut} & \textit{otherwise,} \end{cases} \tag{35}$$

$$\sigma^b := \begin{cases} \sigma^b_{em}\left(\bar{u}_{em}, \underline{u}_{em}\right) & \textit{starting at } t = t^* \textit{ if } h_{t-1} \in H^*_{t-1}, \omega^b = H, \\ \sigma^b_{aut} & \textit{otherwise,} \end{cases} \tag{36}$$

$$\mu^a_t := \begin{cases} 0 & \textit{if } h_{t-1} \notin H^*_{t-1}, \\ 1 & \textit{if } h_{t-1} \in H^*_{t-1} \textit{ and } \exists n < t \textit{ s.t. } \tau_n = (0, 1), \\ \mu_o & \textit{if } h_{t-1} \in H^*_{t-1} \textit{ and } \nexists n \leq t \textit{ s.t. } \tau_n = (z, 0), \\ \mu^a_{t-1} & \textit{if } h_{t-1} \in H^*_{t-1} \textit{ and } w^a_t = 1, \\ \frac{\mu^a_{t-1}(1-2p)}{1-\left(1+\mu^a_{t-1}\right)p} & \textit{otherwise,} \end{cases} \tag{37}$$

$$\begin{aligned} \mu_t^b :=& \quad \begin{cases} 0 & \textit{if } h_{t-1} \notin H_{t-1}^*, \\ 1 & \textit{if } h_{t-1} \in H_{t-1}^* \textit{ and } \exists n < t \textit{ s.t. } \tau_n = (z, 0), \\ \mu_{t-1}^b & \textit{if } h_{t-1} \in H_{t-1}^* \textit{ and } w_t^b = 1, \\ \frac{\mu_{t-1}^b (1-2p)}{1-\left(1+\mu_{t-1}^b\right)p} & \textit{otherwise,} \end{cases} \qquad (38) \\ z \in& \quad [\underline{z}, \bar{z}] \equiv \left[\frac{\mu_o p \delta^L k}{1-\delta^L(1-p)},\ \min\left\{1, \frac{\mu_o p \delta^H\left(1-\delta^H\right)k + \delta^H p(k-1)}{\left(1-\delta^H\right)\left(1-\delta^H(1-2p)\right)}\right\}\right]. \qquad (39) \end{aligned}$$

Then (σ, μ) is a PBE profile.

Proof. Off-equilibrium path beliefs do not need to be consistent in a PBE, so they are moot. During publicly observable equilibrium path play, agent b believes a is a high type with probability 1 if she receives the pre-agreed initial favor z. In the meantime she grows more pessimistic according to Bayesian updating when she receives neither a favor from a nor a favor opportunity during the same period. Her beliefs remain unchanged when she receives the favor opportunity because opportunities are mutually exclusive so there is no new information about the other agent. However, even though agent b grows more pessimistic over time if she does not receive a favor, this does not affect the equilibrium outcome because she is not called upon to do any favors until agent a separates first.

Agent a's beliefs on the other hand remain unchanged until she does the first favor unless there is a public off-equilibrium path action by one of the agents. After the first favor, agent a believes b is a high type once the favor is returned according to (full) EM matching. Until such time her beliefs are updated according to Bayesian updating when she receives neither a favor opportunity, nor a favor from b. Agent a's beliefs remain unchanged when she is the one to receive the favor opportunity. These beliefs are clearly sequentially rational on the equilibrium path so it is enough to solve for z such that the incentive compatibility constraints hold for each agent.

The incentive compatibility constraints obviously hold for agent b since she simply waits until the other does a favor and then plays the EM strategy starting as the disadvantaged agent if she is a high type, and the autarky strategy otherwise. A deviation (favor) on her part would cost her today and result in autarky play.

For a low type agent a we need to solve for the largest size favor, call it $\underline{z}$, she is willing to do to mimic a high type. Similarly we need to solve for the largest size favor, call it $\bar{z}$, a high type agent a is willing to do to signal her type and begin a full EM game as the advantaged agent provided agent b is also a high type. If not, a receives the autarky payoff instead. Bounds $\underline{z}$ and $\bar{z}$ can be computed from the incentive compatibility constraints for the low and high type of agent a, respectively. To end the proof, we show that $[\underline{z}, \bar{z}] \neq \emptyset$. We show that $\underline{z}$ is increasing in δ^L and $\bar{z}$ in δ^H (as well as all other arguments), so using $\delta^L = \delta^H = \delta^*$ we obtain the maximum value for the lower bound and the minimum value for the upper bound, each of which is μ_o. Hence $[\underline{z}, \bar{z}] \neq \emptyset$, and in particular, $\mu_o \in [\underline{z}, \bar{z}]$ for any values of δ^L, δ^H, p and k. The calculations to prove this are essentially straightforward but tedious and have therefore been relegated to the appendix. ■

4.1.1 Characteristics of a designated first favor maker (DFFM) equilibrium

Suppose for the moment that agent a is still, without loss of generality, the designated agent in a DFFM game. If a is a low type, no favors will be exchanged on the equilibrium path, and both agents simply receive their autarky payoff, p. However, if a is a high type facing a low type, her expected payoff would be

$$\begin{aligned} u_D^{HL} &= p\left(\left(1-\delta^H\right)(1{-}z)+\delta^H p\right)+(1-p)\,\delta^H u_D^{HL} \\ &= \tfrac{p\left(\left(1-\delta^H\right)(1-z)+\delta^H p\right)}{1-\delta^H(1-p)} = p - p\tfrac{1-\delta^H}{1-\delta^H(1-p)}z, \end{aligned} \tag{40}$$

and the corresponding payoff for the low type agent b would be

$$\begin{aligned} v_D^{HL} &= p\left(\left(1-\delta^L\right)kz+\delta^L p\right)+p\left(\left(1-\delta^L\right)+\delta^L v_D^{HL}\right)+(1-2p)\,\delta^L v_D^{HL} \\ &= \tfrac{p\left(\left(1-\delta^L\right)(1+kz)+\delta^L p\right)}{1-\delta^L(1-p)} = p + pk\tfrac{1-\delta^L}{1-\delta^L(1-p)}z, \end{aligned} \tag{41}$$

and if agent a is facing another high type, then after the initial favor, her continuation payoff will be $\bar{u}_{em}$ instead of p, so her payoff would be

$$\begin{aligned} u_D^{HH} &= p\left(\left(1-\delta^H\right)(1{-}z)+\delta^H \bar{u}_{em}\right)+(1-p)\,\delta^H u_D^{HH} \\ &= \tfrac{p\left(\left(1-\delta^H\right)(1-z)+\delta^H \bar{u}_{em}\right)}{1-\delta^H(1-p)} = p\tfrac{1-\delta^H+\delta^H\bar{u}_{em}}{1-\delta^H(1-p)} - p\tfrac{1-\delta^H}{1-\delta^H(1-p)}z, \end{aligned}$$

and the corresponding payoff for the high type agent b would be

$$\begin{aligned} v_D^{HH} &= p\left(\left(1-\delta^H\right)kz+\delta^H \underline{u}_{em}\right)+p\left(1-\delta^H+\delta^H v_D^{HH}\right)+(1-2p)\,\delta^H v_D^{HH} \\ &= \frac{p\left(\left(1-\delta^H\right)(1+kz)+\delta^H \underline{u}_{em}\right)}{1-\delta^H\left(1-p\right)} = p\frac{1-\delta^H+\delta^H\underline{u}_{em}}{1-\delta^H(1-p)} + p\frac{\left(1-\delta^H\right)k}{1-\delta^H(1-p)}z. \end{aligned}$$

The following table summarizes the results with payoffs (u, v) for each type combination.

$\omega^a\omega^b$	$u^{\omega^a\omega^b}$	$v^{\omega^a\omega^b}$
LL	p	p
LH	p	p
HL	$p - p\frac{1-\delta^H}{1-\delta^H(1-p)}z$	$p + p\frac{\left(1-\delta^L\right)k}{1-\delta^L(1-p)}z$
HH	$p\frac{1-\delta^H+\delta^H\bar{u}_{em}}{1-\delta^H(1-p)} - p\frac{1-\delta^H}{1-\delta^H(1-p)}z$	$p\frac{1-\delta^H+\delta^H\underline{u}_{em}}{1-\delta^H(1-p)} + p\frac{\left(1-\delta^H\right)k}{1-\delta^H(1-p)}z$

The total value of the game is

$$\begin{aligned} T_D &= (1-\mu_o)\,2p+\mu_o\,(1-\mu_o)\left(u_D^{HL}+v_D^{HL}\right)+\mu_o^2\left(u_D^{HH}+v_D^{H}\right) \\ &= (1-\mu_o)\,2p+\mu_o\,(1-\mu_o)\left(2p - p\tfrac{1-\delta^L}{1-\delta^L(1-p)}z + p\tfrac{\left(1-\delta^L\right)k}{1-\delta^L(1-p)}z\right) \\ &\quad +\mu_o^2\left(p\tfrac{1-\delta^H+\delta^H\bar{u}_{em}}{1-\delta^H(1-p)} + p\tfrac{1-\delta^H+\delta^H\underline{u}_{em}}{1-\delta^H(1-p)} + p\tfrac{\left(1-\delta^H\right)(k-1)}{1-\delta^H(1-p)}z\right). \end{aligned} \tag{42}$$

Substituting in the values for $\bar{u}_{em}$ and $\underline{u}_{em}$ from (4) and (3) and taking the derivative with respect to z, the initial (separating) favor, yields

$$\frac{\partial T_D}{\partial z} \quad = \quad \mu_o p\left((1\text{-}\mu_o)\left(\tfrac{\left(1-\delta^L\right)k}{1-\delta^L(1-p)} - \tfrac{1-\delta^H}{1-\delta^H(1-p)}\right) + \mu_o\tfrac{\left(1-\delta^H\right)(k-1)}{1-\delta^H(1-p)}\right) > 0$$

$$
\begin{aligned}
&\Longleftarrow \quad \left(\tfrac{\left(1-\delta^L\right)k}{1-\delta^L(1-p)} - \tfrac{1-\delta^H}{1-\delta^H(1-p)}\right) > 0 \text{ and other terms clearly positive} \\
&\Longleftarrow \quad \tfrac{1-\delta^L}{1-\delta^L(1-p)} > \tfrac{1-\delta^H}{1-\delta^H(1-p)} \because \delta^L < \delta^H \text{ and } k > 1.
\end{aligned}
$$

In words, the total expected payoff in a DFFM equilibrium is increasing in initial favor size (z). This should not be surprising since doing favors is efficient. Of course, increasing z increases the cost faced by the designee, agent a, to elicit a full EM response from agent b, provided that b is a high type, while a's continuation promise remains unchanged at $u = \mu_o \bar{u}_{em} + (1-\mu_o)\, p$. Naturally, agent b would benefit. But suppose the agents do not know ahead of time whether or not they will be designated to do the first favor, and instead the designation will be determined by a fair coin flip at the beginning of the game. Would the high types want z to be high or low now?

We are interested in this question because each agent who plays this game is either a high type or pretends (at least passively) to be a high type, so any strategies espoused by these agents (pre-separation) should conform to the high types' interests when those interests are unambiguous. For example, if the agents (claiming to be high types) decide to match pennies to decide who is to do the first favor, and if they choose the size of the initial favor $z \in [\underline{z}, \bar{z}]$, they should both purport to favor the optimal z for high types, or else they would reveal their type.

Lemma 14 (Optimal favor size in DFFM equilibria) *Let (σ, μ) be a DFFM equilibrium profile consistent with proposition 13. Then an initial favor of size*

$$
z^i = \begin{cases} \frac{\mu_o \delta^L p k}{1-\delta^L(1-p)} & \textit{if } \mu_o k < 1, \\ \textit{any point in } [\underline{z}, \bar{z}] & \textit{if } \mu_o k = 1, \\ \min\left\{1, \frac{\mu_o \delta^H p\left[\left(1-\delta^H\right)k + \delta^H p(k-1)\right]}{\left(1-\delta^H\right)\left(1-\delta^H(1-2p)\right)}\right\} & \textit{if } \mu_o k > 1, \end{cases} \tag{43}
$$

maximizes expected payoff to a high type agent i provided that the designee is chosen randomly with even odds.

Proof. The expected payoff a high type agent i, who will be designated to do the first favor (that is, to assume the role of agent a of proposition 13) with probability $1/2$, or to wait for the other agent to do so (role b) otherwise, is

$$
\begin{aligned}
u_D^H &= \left(\left(1-\mu_o\right) u_D^{HL} + \mu_o u_D^{HH}\right)/2 + \left(\left(1-\mu_o\right) p + \mu_o v_D^{HH}\right)/2 \\
&= \frac{1}{2}\left[\left(1-\mu_o\right)\left(p - p\tfrac{1-\delta^H}{1-\delta^H(1-p)} z\right) + \mu_o \left(p \tfrac{1-\delta^H+\delta^H \bar{u}_{em}}{1-\delta^H(1-p)} - p\tfrac{1-\delta^H}{1-\delta^H(1-p)} z\right)\right] \\
&\quad + \frac{1}{2}\left[\left(1-\mu_o\right) p + \mu_o \left(p\tfrac{1-\delta^H+\delta^H \underline{u}_{em}}{1-\delta^H(1-p)} + p\tfrac{\left(1-\delta^H\right)k}{1-\delta^H(1-p)} z\right)\right] \\
&= \underbrace{\frac{1}{2} p \left[1-\mu_o + \mu_o \tfrac{1-\delta^H+\delta^H \underline{u}_{em}}{1-\delta^H(1-p)} + 1 - \mu_o + \mu_o \tfrac{1-\delta^H+\delta^H \underline{u}_{em}}{1-\delta^H(1-p)}\right]}_{\equiv \kappa} \\
&\quad + \frac{1}{2} p \left[-\left(1-\mu_o\right) \tfrac{1-\delta^H}{1-\delta^H(1-p)} - \mu_o \tfrac{1-\delta^H}{1-\delta^H(1-p)} + \mu_o \tfrac{\left(1-\delta^H\right)k}{1-\delta^H(1-p)}\right] z
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
&= \kappa + \frac{p\left(1-\delta^H\right)}{2\left(1-\delta^H\left(1-p\right)\right)}\left[-\left(1-\mu_o\right)-\mu_o+\mu_o k\right]z \\
&= \kappa + \frac{p\left(1-\delta^H\right)\left(\mu_o k-1\right)}{2\left(1-\delta^H\left(1-p\right)\right)}z \\
\therefore\quad \frac{\partial u_D^H}{\partial z} &= \frac{p\left(1-\delta^H\right)\left(\mu_o k-1\right)}{2\left(1-\delta^H\left(1-p\right)\right)} \gtreqless 0 \iff \mu_o k \gtreqless 1.
\end{aligned}
$$

Since total welfare of high types is either increasing or decreasing, the optimal choice of the initial favor size is one of the endpoints of interval $[\underline{z}, \bar{z}]$, the set of favors consistent with DFFM equilibria of proposition 13. The only exception is $\mu_o k = 1 \Rightarrow \frac{\partial u_D^H}{\partial z} = 0$, for which case a favor of any size in $[\underline{z}, \bar{z}]$ will do. ■

Lemma 14 applies to DFFM equilibria consistent with proposition 13. That proposition implemented a full EM endgame post-separation. However, we showed in section 3 that high types can always move from the full EM line to the Pareto optimal HSSGL line either immediately, or after sufficiently many periods of EM play. For DFFM equilibria beyond proposition 13, in particular for equilibria that implement a HSSGL endgame, lemma 14 does not apply. However, it stands to reason that since low types would have more incentive to pool with high types, the high types would have more incentive to choose a higher initial favor to deter the low types, and in particular, to cut down the number of periods of EM play necessary to move to the HSSGL endgame in instances where immediate separation to HSSGL is not possible.

4.2 Symmetric separating (SS) equilibria

By symmetric separating equilibria we mean perfect Bayesian equilibria profiles, (σ, μ) , such that, σ^a defines an equivalent strategy for agent a as σ^b does for agent b, and (σ, μ) implements separation with positive probability. We do not require separation with probability one even when both agents are high types. Indeed, we will later show that SS strategies are inherently limited in this regard *when* favor opportunities are correlated. Correlated signals are informative about the other agent's type, which can result in increasingly divergent beliefs between agents over time. And once beliefs diverge sufficiently, low types cannot be deterred from pooling with high types using symmetric strategies. However, we prove that SS equilibria that implement separation with positive probability always exist. First, two definitions.

Definition 15 *Suppose that* (σ, μ) *is a PBE profile for a favor-trading game with two-sided incomplete information such that* σ *implements separation of two high type agents with probability* $(1-\varepsilon) \in (0,1)$. *Call* (σ, μ) *an* ε*-perfect separating PBE.*

Definition 16 (Indicator function) $\mathbf{1}_{\{\mathrm{arg}\}} = 1$ *if* arg *is true,* 0 *otherwise.*

Consider the following example of a $(1-2p)$-perfect separating PBE. Given the opportunity, a high type agent does a (small) favor of size z_1 in the first period. If one of the agents does such a favor, she becomes advantaged in the game of full equality matching that follows provided the other agent is also a high type. If neither agent does a favor in the first period, both agents follow the autarky strategy from thereon. A low type agent follows the autarky strategy regardless. Separation occurs with probability $2p$ if both agents are high types, hence the name $(1-2p)$-perfect separating PBE. This profile is described formally below.

Lemma 17 (First period separation) *Consider strategy profile (σ, μ) defined by (17)-(46) for $i \in \{a, b\}$:*

$$\sigma^i := \begin{cases} I_1^i = z_1 \mathbf{1}_{\{w_1^i = 1\}} & \textit{if } \omega^i = H \textit{ and } t = 1, \\ \sigma_{em}^i \left(\bar{u}_{em}, \underline{u}_{em}\right) & \textit{starting at } t = 2 \textit{ if } \omega^i = H \textit{ and } \tau_1 = \left(z_1, 0\right), \\ \sigma_{em}^i \left(\underline{u}_{em}, \bar{u}_{em}\right) & \textit{starting at } t = 2 \textit{ if } \omega^i = H \textit{ and } \tau_1 = \left(0, z_1\right), \\ \sigma_{aut}^i & \textit{otherwise,} \end{cases} \tag{45}$$

$$\mu_t^i := \begin{cases} 0 & \textit{if } h_{t-1} \notin \mathcal{H}_{t-1}^*, \\ 1 & \begin{array}{l} \textit{if } h_{t-1} \in \mathcal{H}_{t-1}^* \textit{ and either } i \textit{ received } z_1 \\ \quad \textit{in first period or } i \textit{ did the first favor} \\ \quad \textit{and received full reciprocation,} \end{array} \\ \mu_{t-1}^i & \begin{array}{l} \textit{if } h_{t-1} \in \mathcal{H}_{t-1}^* \textit{ and either } i \textit{ received a favor} \\ \quad \textit{opportunity this period, or no favors were done} \\ \quad \textit{in the first period, but } i \textit{ did receive a favor} \\ \quad \textit{opportunity that period (so } \mu_t^i = \mu_o, \forall t\textit{),} \end{array} \\ \frac{\mu_o(1-2p)}{1-(1+\mu_o)p} & \begin{array}{l} \textit{if } h_{t-1} \in \mathcal{H}_{t-1}^*, \textit{ no favors were done and no} \\ \quad \textit{opportunity received by } i \textit{ in first period,} \end{array} \\ \frac{\mu_{t-1}^i(1-2p)}{1-\left(1+\mu_{t-1}^i\right)p} & \textit{otherwise.} \end{cases} \tag{46}$$

Then (σ, μ) is a $(1 - 2p)$-perfect separating PBE profile for $z_1 \in [\underline{z}_1, \bar{z}_1] \neq \emptyset$, where $\underline{z}_1$ and $\bar{z}_1$ are defined by (47).

$$\underline{z}_1 = \frac{\mu_o \delta^L pk}{1-\delta^L(1-p)} \textit{ and } \bar{z}_1 = \min\left\{1, \frac{\mu_o \delta^H \left(\left(1-\delta^H\right)k+\delta^H p(k-1)\right)}{\left(1-\delta^H\right)\left(1-\delta^H(1-2p)\right)}\right\}. \tag{47}$$

Proof. In appendix. ■

The one period case gives a glimpse of how the two-sided incomplete information game behaves. Namely a small favor is necessary to initiate some form of cooperation. The special characteristic of the above equilibrium is that the agents are able to go to full equality matching if both are high types and one of the two initiates cooperation during the first period. Note that as the fraction of high types decreases so does the amount, $\underline{z}_1$, needed to initiate cooperation. This is because both agent types become more pessimistic as μ_o decreases. The high type will be less likely to find a cooperative partner, and the low type will be less likely to find a high type to reciprocate a small initial favor with a full favor. Therefore, it seems likely that in an infinite horizon problem the initial favor necessary to initiate cooperation would decrease as agents grow more pessimistic over time. The problem, however, is that both agents do not necessarily grow pessimistic at the same rate. For example, suppose one agent, say agent a, is a low type and the other, agent b, is a high type. Then it is possible agent a receives a string of favor opportunities at the beginning of the game but does no favors. Agent b would grow more pessimistic by each period, whereas agent a would not receive any information to update her beliefs. At some point the upper bound for the initial favor z derived from a high type's incentive compatibility constraint would cross the lower bound derived from the low type's incentive compatibility constraint for certain histories such as described above with positive probability.

But would it be possible to lower the continuation promise along with z_t to deter the low types? For example, if the EM endgame is conducted at level z_t, we know from lemma 4 that low types would never join in while the high types could benefit from it. However, lemma 4 applies in the complete information environment with two high types, one advantaged, the other disadvantaged right from the start of the game. Without preset designations it turns out that even the high type agents would require a higher continuation promise than just $em\left(z_t\right)$ to start a partnership. In particular, for a game of full equality matching, a high type agent would be at most willing to do an initial favor of $1/2$ for $\delta^H = \delta^*$ and never more than $\frac{k}{k+1}$ even as $\delta^H \rightarrow 1$. The proposition below presents the exact bounds for the general case.

Proposition 18 (Bounds with complete information) *Consider a game of equality matching at level $z_2 \in (0,1]$ between two known high type agents that follows a favor of size z_1 by either agent. The agent to do the initial favor becomes the advantaged agent in the subsequent $em(z_2)$ game. This game can be implemented with any $z_1 \in \left(0, \frac{pk\delta^H}{1+\delta^H(p(k+1)-1)} z_2\right]$ for any $\delta^H \in [\delta^*, 1)$.*

Proof. The advantaged and disadvantaged payoffs, $\bar{u}_{em(z)}$ and $\underline{u}_{em(z)}$, would be the same as before since they are clearly not dependent on any first period payment. The expected payoff for the undesignated agent would be

$$\begin{aligned}\hat{u}_{em(z)} &= p((1-\delta^H)(1-z_1)+\delta^H\bar{u}_{em(z)}) \\ &\quad +p((1-\delta^H)kz_1+\delta^H\underline{u}_{em(z)})+(1-2p)\delta^H\hat{u}_{em(z)}.\end{aligned}$$

Solving for $\hat{u}_{em(z)}$ and substituting in for $\bar{u}_{em(z)}$ and $\underline{u}_{em(z)}$ from (71) and (69) yields

$$\hat{u}_{em(z)} = p + \tfrac{p(1-\delta^H)(k-1)}{1-\delta^H(1-2p)} z_1 + \tfrac{p^2\delta^H(k-1)}{1-\delta^H(1-2p)} z_2.$$

Per proof of lemma 4, we still need $\delta^H \geq \delta^*$ for the incentive compatibility constraints to hold during the second phase of the game that consists of equality matching at level $z_2 \in (0,1]$. The question is, what level of z_1 is required to guarantee that the incentive compatibility constraint is satisfied for the first agent with a favor opportunity to do a favor? As usual, it is enough to consider a one-shot deviation.

$$ICC^H_{first}: \qquad (1-\delta^H)(1-z_1)+\delta^H\bar{u}_{em(z)} \geq 1-\delta^H+\delta^H\hat{u}_{em(z)}.$$

To analyze ICC^H_{first}, let f be a function of the left-hand side minus the right-hand side,

$$\begin{aligned}f &\equiv (1-\delta^H)(1-z_1)+\delta^H\bar{u}_{em(z)}-\left(1-\delta^H\right)-\delta^H\hat{u}_{em(z)} \\ &= -z_1+\delta^H z_1+\delta^H\left(\bar{u}_{em(z)}-\hat{u}_{em(z)}\right) \\ &= -z_1+\delta^H z_1+\delta^H\Big[p+p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}z_2 \\ &\quad -\Big(p+\tfrac{p(1-\delta^H)(k-1)}{1-\delta^H(1-2p)}z_1+\tfrac{p^2\delta^H(k-1)}{1-\delta^H(1-2p)}z_2\Big)\Big] \\ &= -z_1+\delta^H z_1+\tfrac{\delta^H\left(1-\delta^H\right)}{1-\delta^H(1-2p)}p\left(kz_2-\left(k-1\right)z_1\right).\end{aligned}$$

The plot below provides a graphical example of the problem $f \geq 0$, for $z_2 = 1, p = 1/4, k = 5$ fixed, and $z_1 \in [0,1]$ and $\delta^H \in (\delta^*, 1)$ as variables, where $\delta^* = \frac{1}{1+p(k-1)} = \frac{1}{1+\frac{1}{4}(5-1)} = 1/2$.The region above the plane and below the curved surface represents the feasible combinations of z_1 and δ^H that are incentive compatible. The $z_1 = 1$ edge of the graph also shows $f < 0$

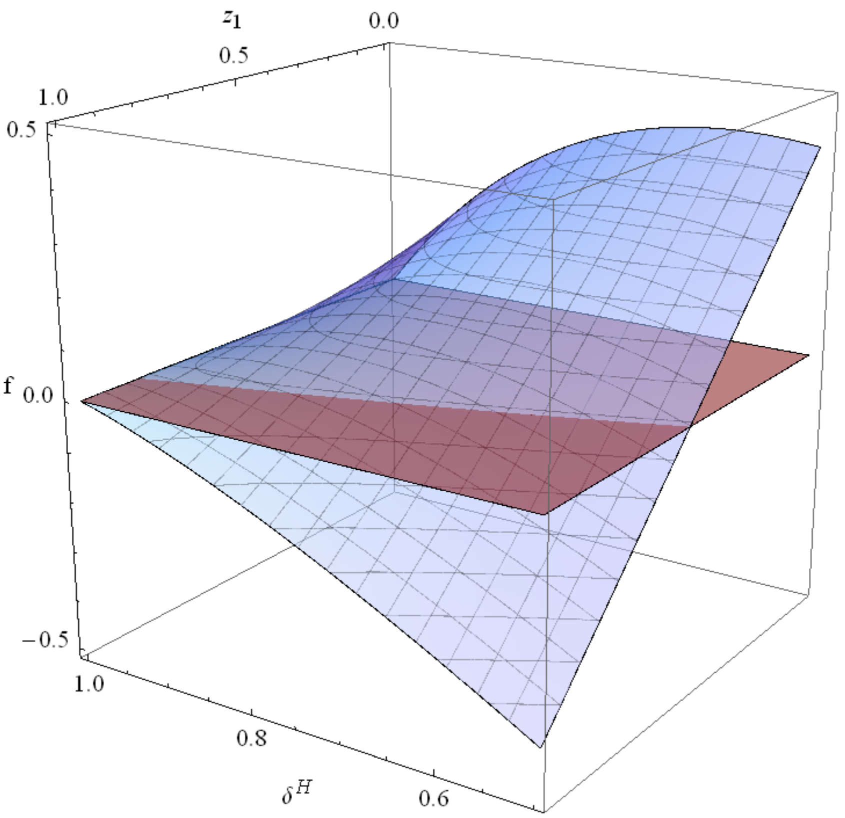

Figure 2: Constraints without designated first mover

(incentive compatibility is not satisfied), for all δ^H, in this case at least. To generalize the intuitions represented in the graph proceed by taking the first two derivatives of f with respect to δ^H,

$$\begin{aligned}\frac{df}{d\delta^H} &= z_1 + \left[\frac{1-\delta^H-\delta^H\left(1-\delta^H(1-2p)\right)}{\left(1-\delta^H(1-2p)\right)^2}\right] p\left(kz_2-(k-1)\,z_1\right), \text{ and}\\ \frac{d^2 f}{d\left(\delta^H\right)^2} &= -\frac{4p}{\left(1-\delta^H(1-2p)\right)^3} p\left(kz_2-(k-1)\,z_1\right) = -\frac{4p^2(kz_2-(k-1)z_1)}{\left(1-\delta^H(1-2p)\right)^3} < 0.\end{aligned}$$

That is, f is strictly concave in δ^H and therefore has a minimum at either $\delta^H = \delta^*$ or $\delta^H = 1$. Evaluating these two limit points of f is a straight-forward exercise that yields the following results after the appropriate simplification,

$$\begin{aligned}\lim_{\delta^H\to 1} f &= 0, \text{ and}\\ \lim_{\delta^H\to\delta^*} f &= -z_1 + \frac{z_1}{1+p(k-1)} + \frac{\frac{1}{1+p(k-1)}\left(1-\frac{1}{1+p(k-1)}\right)}{1-\frac{1-2p}{1+p(k-1)}} p\left(kz_2-(k-1)\,z_1\right)\\ &= \frac{pk(k-1)(z_2-2z_1)}{k+1+p(k^2-1)} \geq 0 \iff z_2 \geq 2z_1.\end{aligned}$$

Therefore a half sized initial favor may always be used by high type agents to commence an equality matching game. To find an upper bound for z_1, say $\bar{z}_1$, we solve $f = 0$, which yields $z_1 = \frac{pk\delta^H}{1+\delta^H(p(k+1)-1)} z_2$. Observe that $\bar{z}_1$ is directly proportional to z_2, and that $\bar{z}_1 < \lim_{\delta^H \to 1} \bar{z}_1 = \frac{k}{k+1} z_2 < z_2$. In summary, a game of equality matching at level $z_2 \in (0,1]$ between high types such that neither agent is designated to go first, but instead the first agent with a favor opportunity is required to do a favor, is incentive compatible for $z_1 \in \left(0, \frac{pk\delta^H}{1+\delta^H(p(k+1)-1)} z_2\right]$, which proves proves proposition 18. ■

If a low type is included in the game among a continuum of high types (that is $\mu_o \approx 1$), using $z_1 < z_2$ presents a clear problem since the original bound for δ^* was derived using $z_1 = z_2$. The ICC for a low type, say agent b, to follow the autarky strategy and not to pool with the high types when $z_1 < z_2$ is as follows:

$$
\begin{array}{rrcl}
ICC^L_{z_1<z_2}: & 1-\delta^L+\delta^L p & \geq & \left(1-\delta^L\right)(1-z_1)+\delta^L\left(\mu_t^b v_{z_2}+(1-\mu_o)\,p\right) \\
\Longrightarrow & \left(1-\delta^L\right) z_1 & \geq & \delta^L\left(v_{z_2}-p\right),
\end{array}
$$

because $\mu_t^b = 1$ since agent b is by assumption the only low type in a continuum of high types. The continuation payoff to b if she does the favor and if agent a is a high type is

$$
\begin{array}{rcl}
v_{z_2} & = & p\left(1-\delta^L+\delta^L v_{z_2}\right)+p\left(\left(1-\delta^L\right)kz_2+\delta^L p\right)+(1-2p)\,\delta^L v_{z_2} \\
& = & p+\frac{pk\left(1-\delta^L\right)}{1-\delta^L(1-p)} z_2.
\end{array}
$$

Substituting from above for v_{z_2} into $ICC^L_{z_1<z_2}$ and simplifying yields

$$
\begin{array}{rcl}
\left(1-\delta^L\right) z_1 & \geq & \delta^L\left(p+\left(1-\delta^L\right)\frac{pk}{1-\delta^L(1-p)} z_2 - p\right) \\
\Longleftrightarrow z_1 & \geq & \delta^L \dfrac{pk}{1-\delta^L(1-p)} z_2 \\
\Longleftrightarrow \delta^L & \leq & \dfrac{1}{1+p\left(k\left(z_2/z_1\right)-1\right)}.
\end{array}
$$

Observe that the bound on δ^L is lower than $\delta^* = 1/\left(1+p\,(k-1)\right)$ for $z_1<z_2$, which identifies an obvious problem with symmetric separating equilibria. Namely, we need to reduce the size of the first favor or limit the time available for separation in order to achieve separation between high types, but this will increase the incentive for low types to mimic the high types unless δ^L is sufficiently low.

Proposition 19 (Time constrained separation in SS equilibria) *Suppose* (σ, μ) *is a PBE profile in a game with two unknown types, and* σ *is such that for* $t \leq T \in \mathbb{N} \cup \{\infty\}$ *the first high type agent to receive a favor opportunity will do a favor of size* $z_t \in [\underline{z}_t, \bar{z}_t] \subset (0,1]$*, where* $[\underline{z}_t, \bar{z}_t]$ *is the interval of incentive compatible first favors at time* t*. If agent* i *does the first favor of size* z_t*,* (σ, μ) *implements an EM strategy for high types, starting with* i *advantaged, at level* $m(z_t)$ *, where* $m : [0,1] \to [0,1]$ *is an increasing function. Then,*

(i) $\exists \bar{n} \in \mathbb{N}$ *such that (a)* $[\underline{z}_t, \bar{z}_t] \neq \emptyset$ *with probability* 1 *for all* $t \leq \bar{n}$*, and (b)* $[\underline{z}_t, \bar{z}_t] = \emptyset$ *with positive probability for all* $t > \bar{n}$*. (Separation between high types is always possible with positive probability, but never with probability* 1 *because it has to occur within a finite time period).*

(ii) In equilibrium, $\bar{z}_t$ and $m(z_t)$ satisfy the following relationship,

$$\frac{m(\bar{z}_t)}{\bar{z}_t} - 1 \leq \frac{1-\delta^L(1-p)}{\mu_o \delta^L pk} \equiv M \in \left(\frac{1}{\mu_o}, \infty\right), \forall t \leq \bar{n}, \tag{48}$$

and separation has to take place within a finite period of time, otherwise a low type may pool with high types with positive probability.

Proof. The details are in the appendix. From lemma 17 we know that $\bar{n} \geq 1$, but we have to show that for t large enough there exist possible histories such that a low type would be willing to do a favor that is equal to or grater than the biggest favor a high type is willing to do. We use a basic proof by contradiction. Suppose a separating symmetric equilibrium exists satisfying (σ, μ) of proposition 19 for $\bar{n} = \infty$. Since neither type can have a profitable one-shot deviation for any history that occurs with positive probability, we proceed by deriving a tight greatest lower bound for z_t from the low type's incentive compatibility constraint. To do this, we use the "worst case" scenario that given t a low type has received favor opportunities each period so far. Consequently the low type's beliefs have remained unchanged at μ_o. This provides us with condition (48), and that is why μ_o is part of the condition. Note that the condition is defined for the upper bounds of z_t and $m(z_t)$ instead of z_t and $m(z_t)$ themselves. That is because we know their upper bounds must be decreasing as agents grow more pessimistic over time, whereas the actual sequence of z_t could behave quite erratically if σ so specified, and the incentive compatibility constraints allowed. We then derive a (slack) upper bound for z_t from the high type's incentive compatibility constraint. This bound has to hold for all z_t in the range of possible equilibrium z_t, so we pick the highest one and show that condition (48) is violated when the high type grows sufficiently pessimistic, that is for t large enough. ■

Corollary 20 *For any equilibrium profile (σ, μ) and sequence $\{z_t\}_{t=1}^{\infty}$ of potential first favors consistent with proposition 19,*

$$m(z_t) = \min\left\{Mz_t + z_t^{LH}, 1\right\} \tag{49}$$

is optimal, where $z_t^{LH} := \left(1 - \delta^L(1-p)\right) \sum_{i=0}^{\infty} \left((1-p)\delta^L\right)^i z_{t+1+i}$.

Proof. This result for $m(z_t)$ is immediate because favors are efficient and $Mz_t + z_t^{LH}$ is the least upper bound on $m(z_t)$ conditional on sequence $z \equiv \{z_t\}_{t=0}^{\infty}$ as specified by (σ, μ). The least upper bound condition is from (104) in the proof of proposition 19. And z_t^{LH} is as specified by (102). ■

Equation (49) may at first seem convoluted because of the inclusion of z_t^{LH}, but that is only the case if z were an infinite sequence of strictly positive terms, which is ruled out by proposition 19. Since the strictly positive terms in z_t are finite, say ending at some $T \in \mathbb{N}$, then $z_T^{LH} = 0$, and consequently $m(z_T) = \min\{Mz_T, 1\}$. We may then calculate the rest of the least upper bounds using backward induction (in theory), although for T large this may be intractable in closed form.

Proposition 21 *For any equilibrium profile* (σ, μ) *consistent with proposition 19 equilibrium separation has to occur within*

$$T \equiv \sup \left\{ t \in \mathbb{N} \,\middle|\, \frac{\mu_o (1-2p)^{t-1}}{\mu_o (1-2p)^{t-1} + (1-\mu_o)(1-p)^{t-1}} \geq \frac{1-\delta^H}{\delta^H BM} \right\} \text{ periods,}$$

where B *and* M *are defined by (71) and (48), respectively.*

Proof. From proposition 19 we know that separation has to occur in finite time, call this T periods at most. Then $z_s = 0$ for all $s>T$, which means that $S_T^{LH} = S_T^{HL} = S_T^{HH} = 0$ for the period T incentive compatibility constraints, which in turn means that we can write condition (107) for T as

$$\begin{aligned} z_T &\leq \frac{\delta^H}{1-\delta^H} \mu_T^a B \, m(z_T) \\ \Longrightarrow \frac{1-\delta^H}{\delta^H B} &\leq \mu_T^a \frac{m(z_T)}{z_T} \leq \mu_T^a M. \end{aligned}$$

Suppose agent a is a high type playing according to σ. Then the history must be such that she has not received a favor opportunity until now and the other agent has done no initial favor. In notation, $(h_T^a, H_{T-1}) = (\{0,0,...,0,1\}, \{0,0,...,0\}) \equiv \left(\widetilde{h}_T^a, \widetilde{H}_{T-1}\right)$. Then $\mu_T^a\left(\widetilde{h}_T^a, \widetilde{H}_{t-1}\right) = \mu_{T-1}^a$ since $w_T^a = 1$, and $\mu_{t-1}^a = \frac{\mu_o(1-2p)^{T-1}}{\mu_o(1-2p)^{T-1}+(1-\mu_o)(1-p)^{T-1}}$ by (109). Substituting this into the above incentive compatibility constraint yields,

$$\frac{1-\delta^H}{\delta^H BM} \leq \frac{\mu_o (1-2p)^{T-1}}{\mu_o (1-2p)^{T-1} + (1-\mu_o)(1-p)^{T-1}}.$$

The left-hand side of the above inequality is just a constant, so we can compute T. Also note that once T is known we can compute z_T from lemma 17, except using μ_T^a instead of μ_o and then work backward to calculate the other z_t. Furthermore, let $g := \frac{1-\delta^H}{\delta^H BM}$, then it is straightforward to show that

$$\frac{\partial g}{\partial \delta^L} = \frac{\mu_o(1-\delta^H)k(1-\delta^H(1-2p))}{\delta^H(1-\delta^L(1-p))^2(k(1-\delta^H(1-p(1-1/k)))} > 0,$$

$$\frac{\partial g}{\partial \delta^H} = -\frac{\mu_o\delta^L k\left((1-\delta^H)^2k+2(1-\delta^H)\delta^H(k-1)p+2\left(\delta^H\right)^2p(k-1)\right)}{\left(\delta^H\right)^2(1-\delta^L(1-p))^2(k(1-\delta^H(1-p(1-1/k)))^2} < 0,$$

so g is maximized at $\delta^L = \delta^H = \delta^*$. Plugging in $\delta^L = \delta^H = \frac{1}{1+p(k-1)}$ and simplifying yields $g|_{\delta^L=\delta^H=\delta^*} = \mu_o$, so the incentive compatibility constraint is always satisfied for at least $T=1$, which we knew already from lemma 17. ■

4.2.1 Importance of mutually exclusive favor opportunities

The proofs of proposition 19 and proposition 21 relied on the mutually exclusive favor opportunities so that one agent could grow pessimistic while the other did not. It would therefore be fair to ask whether symmetric separation has to occur within a fixed time period simply because of the mutually exclusive nature of the favor opportunities and whether independent favor opportunities, which are less informative, could actually lead to a better outcome. The answer, roughly speaking, is that they would. Changing the distribution of favor opportunities from mutually exclusive to independent would change the model drastically since we would then have to worry about favor-trading during periods when both agents received a favor opportunity, which would have a major impact on the incentive compatibility constraints and on the general nature of the problem at hand. However, as a thought experiment, suppose the underlying favor opportunity distribution is kept as is, but it is assumed, contrary to fact, that $\mu_t^a = \mu_t^b = \mu_t$ and that this is common knowledge. Then it can be shown that symmetric separation does not have to occur within a fixed time period.

Proposition 22 (Mutual exclusivity and SS equilibria) *Suppose (σ, μ) is a PBE profile in a game with two unknown types. Suppose σ is such that the first agent to receive a favor opportunity will do a favor of size $z_t \in (0,1]$ to signal type if she is a high type while a low type will follow the autarky strategy. And suppose σ calls for separation to be followed by an equality matching game at some level $m(z_t) \in (0,1]$.Furthermore, suppose agents' beliefs are $\mu_t^a = \mu_t^b = \mu_t$, $\forall t$. Then separation can be guaranteed.*

Proof. From ICC_z^L in the proof of proposition 19 we have that for the low type agents not to mimic the high type agents z_t must satisfy

$$z_t \geq \mu_t^b \frac{\delta^L pk}{1-\delta^L(1-p)} m(z_t),$$

and from ICC_z^H we have that for a high type agent to do the first favor, z_t must satisfy

$$\begin{aligned} z_t \quad \leq \quad & \frac{\delta^H}{1-\delta^H}(1-\mu_t^a) S_1 + \frac{\delta^H}{1-\delta^H}\mu_t^a (Bm(z_t) - S_2), \text{ where} \\ & S_1 = p\left(1-\delta^H\right)\sum_{i=0}^{\infty}\left(\delta^H(1-p)\right)^i z_{t+1+i}, \\ & S_2 = p(k-1)\sum_{i=0}^{\infty}\left(\delta^H(1-2p)\right)^i \left(\left(1-\delta^H\right) z_{t+1+i} + p\delta^H m(z_{t+1+i})\right), \\ & \text{and } B = p\frac{(1-\delta^H)k + \delta^H p(k-1)}{1-\delta^H(1-2p)} \text{ from (71)} \end{aligned}$$

Combining the two conditions above, rearranging and using $\mu_t^a = \mu_t^b = \mu_t$ yields the following condition that must be satisfied as t grow large for separation to be guaranteed:

$$\mu_t \tfrac{\delta^L pk}{1-\delta^L(1-p)} m(z_t) \leq \tfrac{\delta^H}{1-\delta^H}\left((1-\mu_t) S_1 - \mu_t S_2\right) + \tfrac{\delta^H}{1-\delta^H}\mu_t B\, m(z_t). \tag{50}$$

First we need to show that $(1-\mu_t) S_1 - \mu_t S_2 \geq 0$ for t large enough. To this end, observe that

$$\begin{aligned} (1-\mu_t) S_1 - \mu_t S_2 = & (1-\mu_t) p\left(1-\delta^H\right)\sum_{i=0}^{\infty}\left(\delta^H(1-p)\right)^i z_{t+1+i} \\ & -\mu_t p(k-1)\sum_{i=0}^{\infty}\left(\delta^H(1-2p)\right)^i\left(\left(1-\delta^H\right) z_{t+1+i} + p\delta^H m(z_{t+1+i})\right) \end{aligned}$$

$$
\begin{aligned}
&\geq (1-\mu_t)\, p \left(1-\delta^H\right) \sum_{i=0}^{\infty} \left(\delta^H (1-p)\right)^i z_{t+1+i} \\
&\qquad -\mu_t p\, (k-1) \left(1-\delta^H (1-pM)\right) \sum_{i=0}^{\infty} \left(\delta^H (1-p)\right)^i z_{t+1+i} \\
&\qquad \text{since } \left(\delta^H (1-p)\right)^i \geq \left(\delta^H (1-2p)\right)^i, \forall i \geq 0, \\
&\qquad \text{and } m\left(z_{t+1+i}\right) \leq M z_t \text{ by (48)} \\
&= \left((1-\mu_t)\, p \left(1-\delta^H\right) - \mu_t p\, (k-1) \left(1-\delta^H (1-pM)\right)\right) S_{t+1}, \\
&\qquad \text{where } S_{t+1} = \sum_{i=0}^{\infty} \left(\delta^H (1-p)\right)^i z_{t+1+i} \\
&\geq 0 \text{ for } t \text{ large enough } \because \mu_t \to 0 \text{ as } t \to \infty \text{ while} \\
&\qquad p\left(1-\delta^H\right), p\,(k-1)\left(1-\delta^H (1-pM)\right) > 0 \text{ are bounded.}
\end{aligned}
$$

Therefore to verify that condition (50) holds it is enough to verify that

$$
\mu_t \frac{\delta^L pk}{1-\delta^L (1-p)} m\left(z_t\right) \leq \frac{\delta^H}{1-\delta^H} \mu_t B m\left(z_t\right)
$$

or after dividing both sides by $\mu_t m\left(z_t\right)$ and substituting in $B = p\frac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}$ that

$$
\frac{\delta^L pk}{1-\delta^L (1-p)} \leq \frac{\delta^H}{1-\delta^H} p \frac{(1-\delta^H)k+\delta^H p\,(k-1)}{1-\delta^H (1-2p)}.
$$

The left-hand side is clearly increasing in δ^L and $\delta^L < \delta^* = \frac{1}{1+p(k-1)}$ by definition, so the left-hand side is bounded from above by

$$
\frac{\delta^* pk}{1-\delta^* (1-p)} = \frac{\frac{1}{1+p(k-1)} pk}{1-\frac{1}{1+p(k-1)} (1-p)} = 1.
$$

As for the right-hand side, let $h\left(\delta^H\right) = \frac{\delta^H}{1-\delta^H} p \frac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}$. Then $h\left(\delta^*\right) = 1$ and

$$
\begin{aligned}
\frac{dh\left(\delta^H\right)}{d\delta^H} &= \frac{\left(1+(k-1)\,p\right)^2 k}{p\left(k^2-1\right)} > 0 \text{ at } \delta^H = \delta^*, \text{ and} \\
\frac{d^2 h\left(\delta^H\right)}{d\left(\delta^H\right)^2} &= p\left[\frac{k-1}{\left(1-\delta^H\right)^3} + \frac{k+1}{\delta^H\left(1-\delta^H(1-2p)\right)^3} - \frac{k+1}{\delta^H\left(1-\delta^H(1-2p)\right)^2}\right] \\
&> 0 \text{ since } k > 1 \text{ and } 1-\delta^H (1-2p) \in (0,1).
\end{aligned}
$$

In other words, $h\left(\delta^H\right)$ or the right-hand side of our constraint is convex and increasing at its minimum of $h\left(\delta^*\right) = 1$, which was an upper bound for the left-hand side of the constraint. Hence condition (50) is always satisfied for t large enough, and therefore a sequence of z_t would exist such that separation could be guaranteed if the agents believed that their favor opportunities were independent while the game remained exactly as before otherwise. Of course, if the favor opportunities were truly independent then the structure of the game would change significantly because the model uses discrete time and hence we would have to address the possibility that both agents received a favor opportunity in the same round, which would change the incentive compatibility constraints and make it difficult to properly compare the model with mutually exclusive favor opportunities. However, this thought exercise suggests that the impact of non-independent favor opportunities is to make it harder to separate symmetrically. ■

4.2.2 Comparison of SS and DFFM equilibria

Next consider the welfare implications of the SS equilibria as compared to the DFFM equilibria. The appeal of the SS equilibria is that if both agents are high types then separation is twice as fast as in the DFFM equilibrium conditional on it occurring before the agents become too pessimistic. In particular, ignoring a cutoff period for separation, the average times to it, T_{sep}^{ddfm} and T_{sep}^{ss}, can be calculated using geometric series, namely,

$$\begin{aligned} T_{sep}^{ddfm} &= 1p + 2\left(1-p\right)p + 3\left(1-p\right)^{2}p + ... \\ &= p\sum_{t=1}^{\infty} t\left(1-p\right)^{t-1} = \tfrac{1}{p} \end{aligned} \tag{51}$$

and

$$\begin{aligned} T_{sep}^{ss} &= 1(2p) + 2\left(1-2p\right)(2p) + 3\left(1-2p\right)^{2}(2p) + ... \\ &= 2p\sum_{t=1}^{\infty} t\left(1-2p\right)^{t-1} = \tfrac{1}{2p}. \end{aligned} \tag{52}$$

But SS equilibria do not guarantee separation and the value of the post-separation endgame depreciates over time during the initial separation phase unlike with the DFFM equilibria. The expected payoffs for the latter have already been examined, but for the former the expected payoff for a high type at the start of the game would be

$$u_{sym}^{H} = \left(1-\mu_o\right)u_{sym}^{HL} + \mu_o u_{sym}^{HH},$$

where u_{sym}^{HL} and u_{sym}^{H} are the expected payoffs under σ when facing a low type and a high type, respectively. Fortunately u_{sym}^{HL} and u_{sym}^{HH} are symmetric to $u_{-z_t}^{HL}$ and $u_{-z_t}^{HH}$ from the proof of Proposition 19, except that z_t start from z_1 instead of z_{t+1}.

$$\begin{aligned} u_{sym}^{HL} &= p - S_1^{*} \text{ where } S_1^{*} = p\left(1-\delta^{H}\right)\sum_{i=0}^{\infty}\left(\delta^{H}\left(1-p\right)\right)^{i} z_{1+i}, \text{ and} \\ u_{sym}^{HH} &= p + S_2^{*} \text{ where } S_2^{*} = p\left(k-1\right)\sum_{i=0}^{\infty}\left(\delta^{H}\left(1-2p\right)\right)^{i}\left(\left(1-\delta^{H}\right)z_{1+i} + p\delta^{H} m\left(z_{1+i}\right)\right). \end{aligned}$$

In the last step we used that $A + B = p\left(k-1\right)$. Then

$$\begin{aligned} u_{sym}^{H} &= \left(1-\mu_o\right)\left(p - S_1^{*}\right) + \mu_o\left(p + S_2^{*}\right) \\ &= p - S_1^{*} + \mu_o\left(S_1^{*} + S_2^{*}\right). \end{aligned} \tag{53}$$

Next we will go over two examples, the main point of which is that depending on the parameters either SS or DFFM equilibrium may dominate the other.

Example 23 (SS dominates DFFM) *Consider a game with two unknown type agents. Suppose* $p = .45, \delta^L = .6, \delta^H = .8, k = 2, \mu_o = .7$. *Then* $\delta^* = \frac{1}{1+p(k-1)} = .690 \implies \delta^L < \delta^* \leq \delta^H$.

Consider a DFFM equilibrium first and suppose that the agents randomize evenly over who is to be the first favor maker. Since $\mu_o k>1$, this means that z should be maximized per equation (43), so $\bar{z} = \min\left\{1, \frac{\mu_o \delta^H p\left(\left(1-\delta^H\right)k+\delta^H p(k-1)\right)}{\left(1-\delta^H\right)\left(1-\delta^H(1-2p)\right)}\right\} = 1$. And plugging in the values to u_D^H from equation (44) yields $u_D^H = .583$. Assuming the designated first favor maker is a high type, separation will take on average 2.22 rounds but it happens for sure and the endgame will be a full EM game if both agents are high types.

Now compare these results to the SS equilibrium. From proposition 21 we have an upper bound for the number of periods that separation may take

$$T = \sup\left\{t \in \mathbb{N} : \frac{\mu_o (1-2p)^{t-1}}{\mu_o (1-2p)^{t-1} + (1-\mu_o)(1-p)^{t-1}} \geq \frac{1-\delta^H}{\delta^H BM}\right\}.$$

Given the parameters the right-hand side of supremum function is $.3794$ while $\mu_0^a = .7$ and $\mu_1^a = .2979$ so $T = 1$. It is then straightforward to verify that $z = (1, 0, 0, 0, ...)$ maximizes the agents' ex-ante payoffs. Remember that in a pure EM game with no designated first favor maker neither agent would make the first full favor by choice, however, the difference in this example is that if the agent who receives the favor opportunity does not do the favor, then the chance to separate is lost, so in essence she became the disadvantaged agent as soon as she received the favor opportunity, and that is still better than an autarky continuation payoff for the given parameters.

So in this case separation occurs with probability of $.9$ conditional on both agents being high types and will lead to an endgame of full equality matching. The incentive compatibility constraint is satisfied and the expected payoff from (53) is

$$u_{sym}^H = p - p\left(1-\delta^H\right) + \mu_o p\left(\left(1-\delta^H\right) + (k-1)\left(1-\delta^H(1-p)\right)\right) = .599.$$

So in this example the SS equilibrium is clearly better than the best available DFFM equilibrium, which makes sense since separation is $.9$ likely during the first period instead of just $.45$ likely, and the endgames are the same conditional on separation.

Example 24 (DFFM dominates SS) *Consider two agents of unknown type who wish to play a favor-trading game. Suppose* $p = .1, \delta^L = .9, \delta^H = .91, k = 2$ *and* $\mu_o = .45$. *Then* $\delta^* = \frac{1}{1+p(k-1)} = .9091 \implies \delta^L < \delta^* \leq \delta^H$.

Consider a DFFM equilibrium first and suppose that the agents randomize over who is to be the first favor maker. Since $\mu_o k < 1$, this means that z should be minimized per (43), so $\underline{z} = \frac{\mu_o \delta^H pk}{1-\delta^L(1-p)} = .4263$. And plugging in the values to u_D^H from (44) yields $u_D^H = .1103$, which is just above the autarky payoff of $p = .1$. Assuming the designated first favor maker is a high type, separation will take on average 10 rounds but it happens for sure sooner or later and the endgame will be a full EM game if both agents are high types.

Now compare these results to the SS equilibrium. From proposition 21 we have an upper bound for the number of periods that separation may take

$$T = \sup \left\{ t \in \mathbb{N} : \frac{\mu_o (1-2p)^{t-1}}{\mu_o (1-2p)^{t-1} + (1-\mu_o)(1-p)^{t-1}} \geq \frac{1-\delta^H}{\delta^H BM} \right\}.$$

Given the parameters the right-hand side of supremum function $\frac{1-\delta^H}{\delta^H BM} = .4232$ while $\mu_0^a = .45$ and $\mu_1^a = .4211$ assuming no favor opportunity, and no favor was received, so $T = 1$, separation has to occur within the first round of the game if at all. It is then straightforward to verify that $z = (1/M, 0, 0, ...) = (.4263, 0, 0, ...)$ per condition (48) maximizes the agents ex-ante payoff.

So in this case separation occurs with probability of $.2$ conditional on both agents being high types and will lead to an endgame of full equality matching. The incentive compatibility constraint is satisfied and the expected payoff from (53) is

$$u^H_{sym} = p - p\left(1-\delta^H\right) + \mu_o p \left(\left(1-\delta^H\right) + (k-1)\left(1-\delta^H(1-p)\right)\right) = .1037.$$

So in this example the DFFM equilibrium is clearly better than any SS equilibrium, which makes sense since separation is only $.2$ likely in the latter case whereas in the DFFM equilibrium it happens with probability 1 sooner or later and the agents are fairly patient at $\delta^H = .91$.

5 Independent favor opportunities

In this section, unless otherwise stated, we assume favor opportunities arrive independently across agents. In the model so far favor opportunities, w_t^a and w_t^b, were modeled as mutually exclusive. In particular, for $p \in (0, 1/2)$,

$P\left(\left(w_t^a, w_t^b\right) = (1,0)\right) = P\left(\left(w_t^a, w_t^b\right) = (0,1)\right) = p,$
$P\left(\left(w_t^a, w_t^b\right) = (0,0)\right) = (1-2p)$

Because the favor opportunities were correlated they were also informative about the other agent's type. It would be fair to ask whether or not modeling favor opportunities as mutually exclusive was a driving force behind any of the results and what impact this had on the equilibria. To investigate these questions suppose that each agent still receives a favor opportunity with probability $p \in (0, 1)$, but this time the opportunities are independent of each other. For the rest of this section, for $p \in (0, 1)$,

$P\left(\left(w_t^a, w_t^b\right) = (1,1)\right) = p^2$
$P\left(\left(w_t^a, w_t^b\right) = (1,0)\right) = P\left(\left(w_t^a, w_t^b\right) = (0,1)\right) = p(1-p)$
$P\left(\left(w_t^a, w_t^b\right) = (0,0)\right) = (1-p)^2$

Note that the total autarky payoff remains the same as before if p is kept the same $(p < 1/2)$. If neither agent does favors the total average discounted value of the game is $2p$ as before. Similarly, the first best outcome remains unchanged at $2pk$ if both do a favor whenever possible. So have the results changed? The answer to this question with respect to equality matching between two high type agents is no.

Lemma 25 (EM with independent favor opportunities) *In the complete information environment, $\delta^H \geq \delta^*$ is necessary and sufficient to implement any simple EM equilibrium* *that requires the disadvantaged agent to do a favor of size* $x, y \in (0, 1]$, *regardless of whether favor opportunities are independent or mutually exclusive.*

Proof. The proof is essentially the same as in the mutually exclusive case; lemma 4. Below we go over the mathematics for the case of full equality matching $(x = y = 1)$, and explain why the result is the same as with mutually exclusive favor opportunities. A complete proof is available in the appendix.

Suppose the advantaged agent's EM payoff, $\bar{u}'_{em}$, remains unchanged. Then it is immediate that the disadvantaged agent's payoff is also the same as before. If the disadvantaged agent receives a favor opportunity (probability p), she does a full favor in return for a continuation promise of $\bar{u}'_{em}$. If she does not receive a favor opportunity (probability $1-p$), her continuation promise remains $\underline{u}'_{em}$. The equation, $\underline{u}'_{em} = p\delta^H \bar{u}'_{em} + (1-p)\,\delta^H \underline{u}'_{em}$ that defines $\underline{u}'_{em}$ given $\bar{u}'_{em}$ is the same as its counterpart in the mutually exclusive favor opportunities case, equation (1).

The incentive compatibility decision is between forgoing the cost of doing a (full) favor today and receiving the autarky payoff as a continuation promise on the one hand, and doing a full favor today in return for a continuation promise of $\bar{u}'_{em}$. Whether or not the advantaged agent has also received a favor opportunity that same period is clearly of no concern to the disadvantaged agent's utility trade-off decision. The autarky payoff is simply p, as before, and the incentive compatibility constraint has not changed, so if $\bar{u}'_{em}$ is the same as in the game with mutually exclusive favor opportunities, then so is the discount factor required for cooperation; $\delta^H \geq \delta^*$.

Therefore, it is enough to show that the advantaged agent's EM payoff remains unchanged. Writing $\bar{u}'_{em}$ in terms of the four events of the independent favor opportunity distribution yields

$$
\begin{aligned}
\bar{u}'_{em} &= p^2\left(\left(1-\delta^H\right)(k+1)+\delta^H \underline{u}'_{em}\right)+p\,(1-p)\left(1-\delta^H+\delta^H \bar{u}'_{em}\right) \\
&\quad +(1-p)\,p\left(\left(1-\delta^H\right)k+\delta^H \underline{u}'_{em}\right)+(1-p)^2\,\delta^H \bar{u}'_{em}. \\
&= p^2\left(\left(1-\delta^H\right)(k+1)+\delta^H \underline{u}'_{em}\right)+p\left(1-\delta^H+\delta^H \bar{u}'_{em}\right) \\
&\quad -p^2\left(1-\delta^H+\delta^H \bar{u}'_{em}\right)+p\left(\left(1-\delta^H\right)k+\delta^H \underline{u}'_{em}\right) \\
&\quad -p^2\left(\left(1-\delta^H\right)k+\delta^H \underline{u}'_{em}\right)+(1-2p)\,\delta^H \bar{u}'_{em}+p^2\delta^H \bar{u}'_{em} \\
&= p\left(1-\delta^H+\delta^H \bar{u}'_{em}\right)+p\left(\left(1-\delta^H\right)k+\delta^H \underline{u}'_{em}\right)+(1-2p)\,\delta^H \bar{u}'_{em}.
\end{aligned}
$$

In the last step above, we canceled out the p^2-terms to obtain the same equation as in the case with mutually exclusive favor opportunities, equation (2). Therefore, it follows from symmetry that $\bar{u}'_{em} = \bar{u}_{em}$ and $\underline{u}'_{em} = \underline{u}_{em}$, and so it follows that Condition (5) is still the correct bound for cooperation. ■

In subsection 4.2 we already discussed the problem of determining who is advantaged and disadvantaged endogenously. With independent favor opportunities this matter is further complicated by the possibility that both agents receive a favor opportunity in the same period. One solution is formulate the strategy as follows: Agents are *undesignated* at the start of the game. Undesignated agents are called to do a favor of size z. If an agent does such a favor and the other agent does not, the former becomes advantaged and the latter disadvantaged in a game of full equality matching that follows. If both undesignated agents do a favor of size z, both remain undesignated and the stage game is repeated.

Lemma 26 (EM without initial designations) *Consider a complete information environment. Given $z \in (0,1]$, let $t^* := \inf\{s \in \mathbb{N} | \tau_s = (z,0)$ or $(0,z)\}$, denote the first (equilibrium) favor. Let (σ,μ) be defined as follows:*

$$\sigma^i := \begin{cases} I_t^i = z\mathbf{1}_{\{w_t^i=1\}} & \textit{if } h_{t-1} \in \mathcal{H}^*_{t-1}, \omega^i = H \textit{ and } t \leq t^*, \\ \sigma^i_{em}\left(\underline{u}_{em}, \bar{u}_{em}\right) & \textit{from } t = t^*+1 \textit{ if } h_{t-1} \in \mathcal{H}^*_{t-1}, \\ & \quad \omega^i = H \textit{ and } \tau_{t^*} = (0,z), \\ \sigma^i_{em}\left(\bar{u}_{em}, \underline{u}_{em}\right) & \textit{from } t = t^*+1 \textit{ if } h_{t-1} \in \mathcal{H}^*_{t-1}, \\ & \quad \omega^i = H \textit{ and } \tau_{t^*} = (z,0), \\ \sigma^i_{aut} & \textit{otherwise,} \end{cases} \tag{54}$$

$$\mu_t^i := \begin{cases} 0 & \textit{if } h_{t-1} \notin \mathcal{H}^*_{t-1}, \\ 1 & \textit{else if } \exists n \textit{ s.t. for } j \neq i,\ I_n^j = \begin{cases} z \textit{ if } n \leq t^* \\ 1 \textit{ if } n > t^* \end{cases} \\ & \quad \textit{(i.e. other agent has made the initial investment)} \\ \frac{\mu^i_{t-1}(1-p)}{1-\mu^i_{t-1}p} & \textit{otherwise.} \end{cases} \tag{55}$$

Then (σ,μ) is a PBE for $z \leq 1/2$.

Proof. In appendix. A rough outline of the proof and some results are presented below. ■

Once designations are determined, the game will be exactly the same as the one analyzed in lemma 25. In particular, payoffs and incentive compatibility constraints will be the same, so it is enough to evaluate the incentive compatibility constraint for an undesignated agent. Let $\hat{u}^z_{em}$ denote her payoff. Then we can write $\hat{u}^z_{em}$ in terms of the four possible combinations of favor opportunity events:

$$\begin{aligned} \hat{u}^z_{em} &= p^2\left[\left(1-\delta^H\right)\left(1+(k-1)z\right)+\delta^H\hat{u}^z_{em}\right]+ \\ &\quad p(1-p)\left[\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em}\right]+ \\ &\quad (1-p)p\left[\left(1-\delta^H\right)kz+\delta^H\underline{u}_{em}\right]+(1-p)^2\delta^H\hat{u}^z_{em}. \end{aligned}$$

Proof. We already know $\bar{u}_{em}$ and $\underline{u}_{em}$ from the previous lemma, so the above equation only has one unknown, solving for which yields,

$$\begin{aligned} \hat{u}^z_{em} &= p + C_1 + C_2 z, \text{ where} \\ C_1 &\equiv \tfrac{\delta^H(1-p)p^2(k-1)}{1-\delta^H+2\delta^H p(1-p)} \text{ and } C_2 \equiv \tfrac{(1-\delta^H)p(k-1)}{1-\delta^H+2\delta^H p(1-p)} \end{aligned} \tag{56}$$

The incentive compatibility constraint is

$$\begin{aligned} ICC^{nd}_{em} \ &: \ p\left(\left(1-\delta^H\right)\left(1+(k-1)z\right)+\delta^H\hat{u}^z_{em}\right)+(1-p)\left(\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em}\right) \\ &\geq p\left(\left(1-\delta^H\right)(1+kz)+\delta^H\underline{u}_{em}\right)+(1-p)\left(1-\delta^H+\delta^H\hat{u}^z_{em}\right). \end{aligned}$$

When ICC^{nd}_{em} binds we can solve for δ^H. Call the solution,

$$\underline{\delta}^H_z := \frac{2z}{\left(\begin{array}{l} 2z+p\,[p-3z+k(1-z-p(1-2z))+ \\ \sqrt{(p-z)^2+k^2(1-p-z+2pz)^2+2k(1-p-z)(z+p(1-2z))}\,\Big] \end{array}\right)} \tag{57}$$

Alternatively we can solve for z given δ^H :

$$z = \min\left\{1, \frac{\delta^H p(p(1-\delta^H+\delta^H p)+k(1-p-\delta^H(1-3(1-p)p)))}{\left(1-\delta^H+\delta^H(1+k)p-2\delta^H kp^2\right)\left(1-\delta^H(1-2p)\right)}\right\}. \tag{58}$$

Plugging in $\delta^H = \delta^*$, we obtain the bound $z \leq 1/2$. Naturally this is the least upper bound because agents with higher discount factors are willing to pay a higher cost for future benefits. Finally, it is straightforward to verify that beliefs are consistent with σ and Bayesian updating when applicable, so (σ, μ) is a PBE for $z \leq 1/2$. ■

Observe that setting $z = 1/2$ and implementing the strategy profile in lemma 26 is equally efficient to designating one of the do the first favor in a full equality matching game. The former strategy initially implements only half size, but with twice the likely as with full equality matching, so the efficiency gain over autarky in both is $p\,(k-1)$. In particular,

$$\hat{u}_{em}^{1/2} = \tfrac{1}{2}p\,(k+1) = p + \tfrac{1}{2}p\,(k-1)\,. \tag{59}$$

However, for $\delta^H > \delta^*$ we can implement a $z > 1/2$ determined by equation (58), and thereby obtain an efficiency gain over EM with a designated disadvantaged agent from the start. In particular, for $z = 1$, we need

$$\delta^H \geq \underline{\delta}_1^H := \frac{1}{1-2p+p^2\,(k+1)} > \delta^*. \tag{60}$$

Note that (60) cannot be satisfied for any δ^H if $p\,(k+1) < 2$, which roughly speaking rules out cases in which both p and k are high. However, when (60) is satisfied, the initial periods of symmetric favor strategies implement the first best outcome. The next corollary follows by symmetry from the proposition 13 (the equivalent result for the case of mutually exclusive favor opportunities), and is presented mainly as a formality.

Corollary 27 *Let strategy profile (σ, μ) be such that*

$$\sigma^a := \begin{cases} \sigma^a_{em(z)}\left(\underline{u}, \bar{u}\right) & \textit{if } h_{t-1} \in \mathcal{H}^*_{t-1}, \omega^a = H \textit{ and } t = \inf\left\{s \in \mathbb{N} : w^a_s = 1\right\}, \\ \sigma^a_{em}\left(\bar{u}, \underline{u}\right) & \textit{from } t = 1 + \inf\left\{s \in \mathbb{N} : w^a_s = 1\right\} \textit{ if } h_{t-1} \in \mathcal{H}^*_{t-1}, \omega^a = H, \\ \sigma^a_{aut} & \textit{otherwise,} \end{cases} \tag{61}$$

$$\sigma^b := \begin{cases} \sigma^b_{em}\left(\bar{u}, \underline{u}\right) & \textit{from } t = 1 + \inf\left\{s \in \mathbb{N} | \tau_s = (z, 0)\right\} \textit{ if } h_{t-1} \in \mathcal{H}^*_{t-1}, \omega^b = H, \\ \sigma^b_{aut} & \textit{otherwise,} \end{cases} \tag{62}$$

$$\mu^a_t := \begin{cases} 0 & \textit{if } h_{t-1} \notin \mathcal{H}^*_{t-1}, \\ 1 & \textit{if } h_{t-1} \in \mathcal{H}^*_{t-1} \textit{ and } \exists n < t \textit{ s.t. } \tau_n = (0, 1)\,, \\ \mu_o & \textit{if } h_{t-1} \in \mathcal{H}^*_{t-1} \textit{ and } \nexists n \leq t \textit{ s.t. } \tau_n = (z, 0)\,, \\ \frac{\mu^a_{t-1}(1-p)}{1-\mu^a_{t-1}p} & \textit{otherwise,} \end{cases} \tag{63}$$

$$\mu^b_t := \begin{cases} 0 & \textit{if } h_{t-1} \notin \mathcal{H}^*_{t-1}, \\ 1 & \textit{if } h_{t-1} \in \mathcal{H}^*_{t-1} \textit{ and } \exists n < t \textit{ s.t. } \tau_n = (z, 0)\,, \\ \frac{\mu^b_{t-1}(1-p)}{1-\mu^b_{t-1}p} & \textit{otherwise,} \end{cases} \tag{64}$$

$$z \in [\underline{z}, \bar{z}] := \left[\frac{\mu_o \delta^L pk}{1-\delta^L(1-p)}, \min\left\{1, \frac{\mu_o \delta^H p\left[\left(1-\delta^H\right)k + \delta^H p(k-1)\right]}{\left(1-\delta^H\right)\left(1-\delta^H(1-2p)\right)}\right\}\right]. \tag{65}$$

Then (σ,μ) *is a PBE profile.*

Proof. Immediate from symmetry with proposition 13. ■

Proposition 28 (SS equilibria and independent arrivals) *Let* $t^*{:=}\inf\{s{\in}\mathbb{N} : \tau_s = (z_s, z_s), (z_s, 0)$ *or* $(0, z_s)\}$ *where* $\inf\{\emptyset\} \equiv \infty$, *be the time of the first favor, and let* (σ,μ) *be defined as follows for* $i \in \{a, b\}$:

$$\sigma^i := \begin{cases} I_t^i = z_t \mathbf{1}_{\{w_t^i=1\}} & \text{if } h_{t-1} \in \mathcal{H}^*_{t-1} \text{ and } \omega^i = H, t{<}t^* \\ & \text{(no one has done a favor yet),} \\ & \text{starting at } t = t^*+1 \text{ if } \tau_{t^*} \text{ symmetric and} \\ I_t^i = \mathbf{1}_{\{w_t^i=1\}} & \text{ending at } n{>}t^*+1 \text{ s.t. } \tau_{n-1} \neq (1,1) \text{ and} \\ & \text{only if } h_{t-1} \in \mathcal{H}^*_{t-1} \text{ and } \omega^i = H, \\ \sigma^i_{em}\left(\bar{u}_{em}, \underline{u}_{em}\right) & \text{starting after first non-symmetric } \tau_{t-1} \text{ if } a \text{ did the} \\ & \text{favor, and only if } h_{t-1} \in \mathcal{H}^*_{t-1} \text{ and } \omega^i = H, \\ \sigma^i_{em}\left(\underline{u}_{em}, \bar{u}_{em}\right) & \text{starting after first non-symmetric } \tau_{t-1} \text{ if } b \text{ did the} \\ & \text{favor, and only if } h_{t-1} \in \mathcal{H}^*_{t-1} \text{ and } \omega^i = H, \\ \sigma^i_{aut} & \text{otherwise,} \end{cases} \tag{66}$$

$$\mu_t^i := \begin{cases} 0 & \text{if } h_{t-1} \notin H^*_{t-1}, \\ 1 & \text{if } h_{t-1} \in H^*_{t-1}, t > t^* \text{ and } j \neq i \text{ did the} \\ & \text{first favor or reciprocated,} \\ \frac{\mu^i_{t-1}(1-p)}{1-\mu^i_{t-1}p} & \text{otherwise.} \end{cases} \tag{67}$$

Then if δ^H *satisfies condition (60), there exists a sequence* $z = \{z_s\}_{s=0}^{\infty} \in (0,1]^{\infty}$ *such that* (σ,μ) *is a separating PBE profile. If both agents are high types,* σ *will implement separation with probability one.*

Proof. Details in appendix. ■

6 Conclusion

We have shown that using a simple EM mechanism, immediate separation can be implemented in the favor-trading games with one-sided incomplete information with only the minimal requirement that $\delta^L < \delta^*$ for the low types. We also describe necessary conditions for immediate separation into more profitable equilibria, and a workable strategy to reach them over time if an immediate move is not possible.

In the case of two-sided incomplete information, separation can be guaranteed with probability 1 if one of the agents is designated to be the first favor maker at the beginning of the game. However, in numerical testing symmetric equilibria with no designated first favor maker tended to perform better even though these equilibria have the unfortunate characteristic that separation has to take place within a fixed period at the beginning of the game before the gap between potential beliefs about each other becomes too wide. With independent favor opportunities beliefs remain symmetric until someone does a favor, and therefore separation under symmetric equilibria is not limited to a finite period.

7 Appendix

Proof. (Lemma 7: Necessary and sufficient condition for separation under one-sided incomplete information)

Suppose we start the game with agent b as the advantaged agent, and the level of trust is $z \in (0,1]$; that is agent a does a favor of size $x = z$ if she receives a favor opportunity, and does no further favors until the other agent, in this case agent b, reciprocates by doing a favor of size $y = z$. It follows that in terms of average discounted payoffs

$$\begin{aligned}
\underline{u}_{em(z)} &= p\left(\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em(z)}\right)+(1-p)\,\delta^H\underline{u}_{em(z)} \\
&= p\tfrac{\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em(z)}}{1-\delta^H(1-p)}. \\
\bar{u}_{em(z)} &= p\left(\left(1-\delta^H\right)+\delta^H\bar{u}_{em(z)}\right) \\
&\quad +p\left(\left(1-\delta^H\right)kz+\delta^H\underline{u}_{em(z)}\right)+(1-2p)\,\delta^H\bar{u}_{em(z)} \\
&= p\tfrac{\left(1-\delta^H\right)(1+kz)+\delta^H\underline{u}_{em(z)}}{1-\delta^H(1-p)}.
\end{aligned}$$

The two equations above are in two unknowns, $\underline{u}_{em(z)}$ and $\bar{u}_{em(z)}$. Solving for these yields,

$$\begin{aligned}
\underline{u}_{em(z)} &= p+p\tfrac{-\left(1-\delta^H\right)+\delta^H p(k-1)}{1-\delta^H(1-2p)}z \qquad (68) \\
&= p+Az \text{ where } A \equiv p\tfrac{-\left(1-\delta^H\right)+\delta^H p(k-1)}{1-\delta^H(1-2p)} \qquad (69) \\
\bar{u}_{em(z)} &= p+p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}z \qquad (70) \\
&= p+Bz \text{ where } B \equiv p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}. \qquad (71)
\end{aligned}$$

For z fixed, as δ^H ranges from δ^* to 1, $\underline{u}_{em(z)}$ ranges from p to$p+\frac{1}{2}p\,(k-1)\,z$ and $\bar{u}_{em(z)}$ from $p+p(k-1)z$ to $p+\frac{1}{2}p\,(k-1)\,z$. In particular, for any $\delta^H \in [\delta^*, 1)$,

$$\begin{aligned}
\underline{u}_{em(z)}+\bar{u}_{em(z)} &= 2p+p(k-1)z, \text{ or} \\
&= p\,(k+1) \text{ for } z=1. \qquad (72)
\end{aligned}$$

Agent a's incentive compatibility constraint is

$$\begin{aligned}
ICC^a_{em(z)} : \left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em(z)} &\geq \left(1-\delta^H\right)+\delta^H\underline{u}_{em(z)} \\
\iff \bar{u}_{em(z)}-\underline{u}_{em(z)}-\tfrac{\left(1-\delta^H\right)}{\delta^H}z &\geq 0,
\end{aligned}$$

which using equations (68) and (70) is equivalent to

$$\begin{aligned}
&p+p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}z-p-p\tfrac{-\left(1-\delta^H\right)+\delta^H p(k-1)}{1-\delta^H(1-2p)}z \geq \tfrac{\left(1-\delta^H\right)}{\delta^H}z \\
&\text{solving for } \delta^H \implies \tfrac{1}{1+p(k-1)} \leq \delta^H.
\end{aligned}$$

Recall that $\delta^* = \dfrac{1}{1+p\,(k-1)}$ so $\delta^* \leq \delta^H$ is necessary and sufficient to implement any simple EM strategy profile. $x = y = 1$ ensures the greatest gains from cooperation. ■

Proof. (Lemma 8: HSSGL payoff to advantaged low types)

Let $\overline{v}_s^L$ denote agent b's continuation payoff s periods after the game starts conditional on b not having received a favor yet. Let $\underline{v}_s^L$ denote agent b's continuation payoff s periods after she has received a favor. And let $\bar{x}_s$ and $\underline{x}_s$ denote the favors as specified by (16) and (17) that agent a will do following states assoiciaated with $\overline{v}_s^L$ and $\underline{v}_s^L$, respectively. From equation (11) we already know that $\overline{x}_0 = 1$ and $\underline{x}_0 = \frac{\delta^H - \delta^*}{\delta^H + \delta^*}$. Given the notation, we may write agent b's expected payoff as follows,

$$\begin{aligned}
\overline{v}_0^L &= p\left(\left(1-\delta^L\right)k\overline{x}_0 + \delta^L \underline{v}_0^L\right) + (1-p)\left(\left(1-\delta^L\right)\tfrac{p}{1-p} + \delta^L \overline{v}_1^L\right) \\
&= p\left(1-\delta^L\right) + p\delta^L \underline{v}_0^L + p\left(1-\delta^L\right)k\overline{x}_0 + (1-p)\,\delta^L \overline{v}_1^L \\
&= p\left(1-\delta^L\right) + p\delta^L \underline{v}_0^L + p\left(1-\delta^L\right)k\overline{x}_0 \\
&\quad + (1-p)\,\delta^L\left(p\left(1-\delta^L\right) + p\delta^L \underline{v}_0^L + p\left(1-\delta^L\right)k\overline{x}_1 + (1-p)\,\delta^L \overline{v}_2^L\right) \\
&= p\left(1-\delta^L\right)\left(1+(1-p)\,\delta^L\right) + p\delta^L \underline{v}_0^L\left(1+(1-p)\,\delta^L\right) \\
&\quad + p\left(1-\delta^L\right)k\left(\overline{x}_0 + (1-p)\,\delta^L \overline{x}_1\right) + \left((1-p)\,\delta^L\right)^2 \overline{v}_2^L \\
&= p\left(1-\delta^L\right)\left(1+d+d^2+\ldots\right) + p\delta^L \underline{v}_0^L\left(1+d+d^2+\ldots\right) \\
&\quad + p\left(1-\delta^L\right)k\left(\overline{x}_0 + d\overline{x}_1 + d^2\overline{x}_2 + \ldots\right) \text{ where } d = (1-p)\,\delta^L \\
&= p\tfrac{1-\delta^L}{1-d} + \tfrac{p\delta^L}{1-d}\underline{v}_0^L + p\left(1-\delta^L\right)k\sum_{t=0}^{\infty} d^t \overline{x}_t.
\end{aligned} \tag{73}$$

To proceed further, we need to calculate $\underline{v}_0^L$ and $\overline{x}_t$.

Claim 8a: Let $\bar{u}(t)$ denote the HSSGL continuation promise to an advantaged agent t periods since the last favor in a game of complete information. Then,

$$\bar{u}(t) = p + \frac{\left(\delta^H\right)^{t+1}(1-\delta^*) + \left(1-\delta^H\right)(\delta^*)^{t+1}}{\left(\delta^H\right)^t \delta^*\left(\delta^H+\delta^*\right)}, \tag{74}$$

$$\overline{x}_t = \frac{\delta^H + \left(\delta^*/\delta^H\right)^t \delta^*}{\delta^H + \delta^*}. \tag{75}$$

Proof of claim 8a: By (14) and (15),

$$\begin{aligned}
\bar{u} &= \underline{u} + \frac{2\left(1-\delta^H\right)}{\delta^H+\delta^*} = p + \frac{\delta^H - \delta^*}{\delta^H + \delta^*}\frac{1}{\delta^*} + \frac{2\left(1-\delta^H\right)}{\delta^H+\delta^*} \\
&= p + \frac{\delta^H(1-\delta^*) + \left(1-\delta^H\right)\delta^*}{\delta^*\left(\delta^H+\delta^*\right)} = \bar{u}(0).
\end{aligned}$$

Suppose (74) holds for some $s \in \mathbb{N}$, then by (17),

$$\begin{aligned}
\bar{u}(s+1) &= \delta^*\left(\frac{\bar{u}(s) - \left(1-\delta^H\right)p}{\delta^H} + \frac{\underline{u}(1-\delta^*)}{\delta^*}\right) \\
&= \delta^*\left(\frac{1}{\delta^H}\left(p + \frac{\left(\delta^H\right)^{s+1}(1-\delta^*) + \left(1-\delta^H\right)(\delta^*)^{s+1}}{\left(\delta^H\right)^s \delta^*\left(\delta^H+\delta^*\right)}\right) - p\frac{1-\delta^H}{\delta^H} + \left(p + \frac{\delta^H-\delta^*}{\delta^H+\delta^*}\frac{1}{\delta^*}\right)\frac{1-\delta^*}{\delta^*}\right) \\
&= \delta^*\left(\frac{1}{\delta^H} - \frac{1-\delta^H}{\delta^H} + \frac{1-\delta^*}{\delta^*}\right)p + \frac{\delta^*}{\delta^H}\frac{\left(\delta^H\right)^{s+1}(1-\delta^*) + \left(1-\delta^H\right)(\delta^*)^{s+1}}{\left(\delta^H\right)^s \delta^*\left(\delta^H+\delta^*\right)} + (1-\delta^*)\frac{\delta^H-\delta^*}{\delta^H+\delta^*}\frac{1}{\delta^*}
\end{aligned}$$

$$= \quad p+\frac{\left(\delta^H\right)^{s+2}(1-\delta^*)+\left(1-\delta^H\right)(\delta^*)^{s+2}}{\left(\delta^H\right)^{s+1}\delta^*\left(\delta^H+\delta^*\right)}=\bar{u}\left(s+1\right).$$

Given that equation (74) holds for $s+1$ if it holds for s, and we know it holds for $s=0$, then by induction (74) must hold for all $s\in\mathbb{N}$. Last, it is a straightforward computation to verify that if we apply (16), the equation to compute HSSGL favors from payoffs, to (74), the hypothetical payoff to a high type, and simplify, formula (75) results. [■]

Claim 8b: Let $\underline{u}\left(t\right)$ denote the HSSGL continuation promise to a disadvantaged agent t periods since the last favor in the complete information game. Then,

$$\underline{u}\left(t\right)=p+\frac{\left(\delta^H\right)^{t+1}(1-\delta^*)-\left(1-\delta^H\right)(\delta^*)^{t+1}}{\left(\delta^H\right)^{t}\delta^*\left(\delta^H+\delta^*\right)} \tag{76}$$

$$\underline{x}_t=\frac{\delta^H-\left(\delta^*/\delta^H\right)^t\delta^*}{\delta^H+\delta^*} \tag{77}$$

Proof of claim 8b: By (14), $\underline{u}=p+\frac{\delta^H-\delta^*}{\delta^H+\delta^*}\frac{1}{\delta^*}=\underline{u}\left(0\right)$. Suppose (76) holds for some $s\in\mathbb{N}$, then by (17)

$$\begin{aligned}
\underline{u}\left(s+1\right) &= \delta^*\left(\frac{\underline{u}(s)-\left(1-\delta^H\right)p}{\delta^H}+\frac{\underline{u}(1-\delta^*)}{\delta^*}\right)\\
&= \frac{\delta^*}{\delta^H}\left(p+\frac{\left(\delta^H\right)^{s+1}(1-\delta^*)-\left(1-\delta^H\right)(\delta^*)^{s+1}}{\left(\delta^H\right)^{s}\delta^*\left(\delta^H+\delta^*\right)}\right)-\frac{1-\delta^H}{\delta^H}p+\left(p+\frac{\delta^H-\delta^*}{\delta^H+\delta^*}\frac{1}{\delta^*}\right)\frac{1-\delta^*}{\delta^*}\\
&= \delta^*\left(\frac{1}{\delta^H}-\frac{1-\delta^H}{\delta^H}+\frac{1-\delta^*}{\delta^*}\right)p+\frac{\delta^*}{\delta^H}\frac{\left(\delta^H\right)^{s+1}(1-\delta^*)-\left(1-\delta^H\right)(\delta^*)^{s+1}}{\left(\delta^H\right)^{s}\delta^*\left(\delta^H+\delta^*\right)}+\frac{\delta^H-\delta^*}{\delta^H+\delta^*}\frac{1-\delta^*}{\delta^*}\\
&= p+\frac{\left(\delta^H\right)^{s+2}(1-\delta^*)+\left(1-\delta^H\right)(\delta^*)^{s+2}}{\left(\delta^H\right)^{s+1}\delta^*\left(\delta^H+\delta^*\right)}.
\end{aligned}$$

Given that (76) holds for $s+1$ if it holds for s, and we know it holds for $s=0$, then by induction (76) must hold for all $s\in\mathbb{N}$. Last, it is a straightforward computation to verify that if we apply (16), the equation to calculate favors from payoffs, to (76), the hypothetical payoff to a high type, and simplify, and simplify, formula (77) results. [■]

Claim 8c: Let $d=(1-p)\,\delta^L$, then

$$\underline{v}_0^L=p+pk\left(1-d\right)\sum_{t=0}^{\infty}d^t\underline{x}_t \tag{78}$$

Proof of claim 8c: Proceeding as before.

$$\begin{aligned}
\underline{v}_0^L &= p\left(\left(1-\delta^L\right)k\underline{x}_0+\delta^L\underline{v}_0^L\right)+\left(1-p\right)\left(\left(1-\delta^L\right)\frac{p}{1-p}+\delta^L\underline{v}_1^L\right)\\
&= p\left(1-\delta^L\right)+p\delta^L\underline{v}_0^L+p\left(1-\delta^L\right)k\underline{x}_0+\left(1-p\right)\delta^L\underline{v}_1^L\\
&= p\left(1-\delta^L\right)+p\delta^L\underline{v}_0^L+p\left(1-\delta^L\right)k\underline{x}_0\\
&\quad+\left(1-p\right)\delta^L\left(\left(1-\delta^L\right)p+p\delta^L\underline{v}_0^L+p\left(1-\delta^L\right)k\underline{x}_1+\left(1-p\right)\delta^L\underline{v}_2^L\right)\\
&= p\left(1-\delta^L\right)\left(1+\left(1-p\right)\delta^L\right)+p\delta^L\underline{v}_0^L\left(1+\left(1-p\right)\delta^L\right)\\
&\quad+p\left(1-\delta^L\right)k\left(\underline{x}_0+\left(1-p\right)\delta^L\underline{x}_1\right)+\left(\left(1-p\right)\delta^L\right)^2\underline{v}_2^L
\end{aligned}$$

$$\begin{aligned}
&= p\left(1-\delta^L\right)\left(1+d+d^2+...\right)+p\delta^L \underline{v}_0^L\left(1+d+d^2+...\right) \\
&\quad +p\left(1-\delta^L\right)k\left(\underline{x}_0+d\underline{x}_1+d^2\underline{x}_2+...\right) \text{ where } d=(1-p)\,\delta^L \\
&= \tfrac{p\left(1-\delta^L\right)}{1-d}+\tfrac{p\delta^L}{1-d}\underline{v}_0^L+p\left(1-\delta^L\right)k\sum_{t=0}^{\infty} d^t \underline{x}_t \\
&= p+pk\,(1-d)\sum_{t=0}^{\infty} d^t \underline{x}_t
\end{aligned}$$

This proves (78). [■]
We can now return to equation (73) for $\overline{v}_0^L$ and substituting in for $\underline{v}_0^L$ from (78) yields,

$$\begin{aligned}
\overline{v}_0^L &= \tfrac{p\left(1-\delta^L\right)}{1-d}+\tfrac{p\delta^L}{1-d}\left(p+pk\,(1-d)\sum_{t=0}^{\infty} d^t \underline{x}_t\right)+p\left(1-\delta^L\right)k\sum_{t=0}^{\infty} d^t \overline{x}_t \\
&= \tfrac{p\left(1-\delta^L\right)}{1-d}+\tfrac{p^2\delta^L}{1-d}+p^2k\delta^L\sum_{t=0}^{\infty} d^t \underline{x}_t+p\left(1-\delta^L\right)k\sum_{t=0}^{\infty} d^t \overline{x}_t \\
&= p+pk\sum_{t=0}^{\infty} d^t\left(p\delta^L\underline{x}_t+\left(1-\delta^L\right)\overline{x}_t\right)
\end{aligned}$$

Using (75) and (77) for $\overline{x}_t$ and $\underline{x}_t$, respectively, we can write the last equation as

$$\begin{aligned}
\overline{v}_0^L &= p+pk\sum_{t=0}^{\infty} d^t\left(p\delta^L\tfrac{\delta^H-\left(\delta^*/\delta^H\right)^t\delta^*}{\delta^H+\delta^*}+\left(1-\delta^L\right)\tfrac{\delta^H+\left(\delta^*/\delta^H\right)^t\delta^*}{\delta^H+\delta^*}\right) \\
&= p+pk\sum_{t=0}^{\infty} d^t\tfrac{\left(\delta^H\right)^{t+1}\left(1-\delta^L(1-p)\right)+(\delta^*)^{t+1}\left(1-\delta^L-\delta^L p\right)}{\left(\delta^H\right)^t\left(\delta^H+\delta^*\right)} \\
&= p+pk\sum_{t=0}^{\infty} d^t\tfrac{1-\delta^L(1-p)+\alpha^{t+1}\left(1-\delta^L-\delta^L p\right)}{1+\alpha} \text{ where } \alpha=\tfrac{\delta^*}{\delta^H} \\
&= p+pk\tfrac{1-\delta^L(1-p)}{(1+\alpha)(1-d)}+pk\tfrac{\alpha\left(1-\delta^L-\delta^L p\right)}{(1+\alpha)(1-\alpha d)}.
\end{aligned}$$

Next we substitute $d=(1-p)\,\delta^L$ and $\alpha=\delta^*/\delta^H$ back in,

$$\begin{aligned}
\overline{v}_0^L &= p+pk\tfrac{1-\delta^L(1-p)}{\left(1+\delta^*/\delta^H\right)\left(1-(1-p)\delta^L\right)}+pk\tfrac{\left(\delta^*/\delta^H\right)\left(1-\delta^L-\delta^L p\right)}{\left(1+\delta^*/\delta^H\right)\left(1-\left(\delta^*/\delta^H\right)(1-p)\delta^L\right)} \\
&= p+pk\tfrac{\delta^H\left(\delta^H+\delta^*-2\delta^L\delta^*\right)}{\left(\delta^H+\delta^*\right)\left(\delta^H-\delta^L\delta^*(1-p)\right)}.
\end{aligned}$$

This concludes the proof. ■

Proof. (Corollary 12: From EM to HSSGL) If a low type does a full favor $y=1,$ then for the next T periods it will be optimal for her to play the autarky strategy instead of exchanging favors according to the simple EM mechanism as was shown in section 2. Therefore we may ignore favor opportunities she receives during those T periods as far as the incentive compatibility constraint is concerned because she would have received these opportunities had she chosen not to deviate by doing a favor of size $y=1$. However, we do have to calculate the expected amount she will receive in reciprocation from agent a. Namely, in the first period after separation agent

a will receive a favor opportunity with probability p and do a full favor worth k, but she will not do any further favors during the rest of the T periods since agent b does not reciprocate. With probability $(1-p)\,p$ agent a will not receive a favor opportunity during period 1, but will do so in period 2, and thus does a full favor, but no more until reciprocation for the rest of the T periods. And so forth for the other T periods. Let v^L be her expected payoff from deviating minus the (autarky) favor costs saved during the first T periods,

$$\begin{aligned} v^L &= \left(1-\delta^L\right)\Big(pk+\delta^L\left(1-p\right)p+\left(\delta^L\right)^2\left(1-p\right)^2 pk+\ldots \\ &\quad +\left(\delta^L\right)^{T-1}\left(1-p\right)^{T-1}pk\Big)+\left(\delta^L\right)^T\left(1-p\right)^T\overline{v}^L+\left(\delta^L\right)^T\Big(1-\left(1-p\right)^T\Big)\underline{v}^L, \end{aligned}$$

where the $\left(\delta^L\right)^T$-terms are the continuation payments after T periods pass. With probability $(1-p)^T$ agent a did not receive income during any of the T periods and was thus not able to reciprocate which is why agent b remains advantaged and receives the continuation promise $\overline{v}^L$. Otherwise a has reciprocated and is currently the advantaged agent, so b's continuation promise is $\underline{v}^L$.

Note that the above equation contains a geometric series that can be written more compactly, and using the fact that $\underline{v}^L \leq \overline{v}^L$ we know

$$\begin{aligned} v^L &\leq \left(1-\delta^L\right)pk\sum_{t=0}^{T-1}\left(\delta^L-\delta^L p\right)^t+\left(\delta^L\right)^T\left(1-p\right)^T\overline{v}^L+\left(\delta^L\right)^T\overline{v}^L-\left(\delta^L\right)^T\left(1-p\right)^T\overline{v}^L \\ &= pk\frac{\left(1-\delta^L\right)\left(1-\left(\delta^L-\delta^L p\right)^T\right)}{1-\delta^L(1-p)}+\left(\delta^L\right)^T\overline{v}^L. \qquad (79) \end{aligned}$$

The incentive compatibility constraint for the low type not to pool is $\delta^L v^L \leq 1-\delta^L+\left(\delta^L\right)^{T+1}p$. Note that both sides of the inequality exclude T-terms after the initial favor by agent b since they cancel each other out. Then according to condition (79) it is sufficient to show that

$$\delta^L\frac{\left(1-\delta^L\right)pk\left(1-\left(\delta^L-\delta^L p\right)^T\right)}{1-\delta^L(1-p)}+\left(\delta^L\right)^{T+1}\overline{v}^L \leq 1-\delta^L+\left(\delta^L\right)^{T+1}p.$$

Since $\left(1-\left(\delta^L-\delta^L p\right)^T\right)<1$ and $\overline{v}^L<p\left(k+1\right)$ by (26) from the proof of lemma 10 it is enough to show that

$$\begin{aligned} & \delta^L\frac{\left(1-\delta^L\right)pk}{1-\delta^L(1-p)}+\left(\delta^L\right)^{T+1}p\left(1+k\right) \leq \left(1-\delta^L\right)+\left(\delta^L\right)^{T+1}p \\ \iff\quad & \delta^L\frac{\left(1-\delta^L\right)pk}{1-\delta^L(1-p)} \leq 1-\delta^L-\left(\delta^L\right)^{T+1}pk \\ \iff\quad & \delta^L\frac{pk}{1-\delta^L(1-p)} \leq 1-\frac{\left(\delta^L\right)^{T+1}}{1-\delta^L}pk. \qquad (80) \end{aligned}$$

To enforce the inequality we need to construct a T sufficiently high that it holds. To this end observe that since $\delta^L<\delta^*$ there exists $\varepsilon>0$ such that

$$\delta^L\frac{pk}{1-\delta^L(1-p)}+\varepsilon=\delta^*\frac{pk}{1-\delta^L(1-p)}=\frac{pk}{\left(1+p(k-1)\right)\left(1-\delta^L(1-p)\right)}.$$

Choose T to be the least integer such that $pk\left(\delta^L\right)^{T+1}/\left(1-\delta^L\right)\leq\varepsilon/2$. Simplifying the last expression, it is straightforward to show that

$$\frac{pk}{(1+p(k-1))\left(1-\delta^L(1-p)\right)} < \frac{pk}{(1+p(k-1))(1-\delta^*(1-p))} = 1,$$

so we have that the left-hand side of (80) is strictly than $1-\varepsilon$ and the right-hand side is greater than $1-\varepsilon/2,$ therefore (80) holds, which implies that the incentive compatibility constraint is satisfied and therefore the low type will not pool for this choice of T. ■

Proof. (Simple lower bound for $\bar{\delta}^L$)

Suppose agent b is a low type, then b's incentive compatibility constraint to do the first favor is

$$ICC^L_{hssgl} : \delta^L \overline{v}^L_0 \geq \left(1-\delta^L\right) + \delta^L p$$

Substituting in for $\overline{v}^L_0$ from equation (22) yields

$$\delta^L \left(p + pk \frac{\delta^H\left(\delta^H+\delta^*-2\delta^L\delta^*\right)}{\left(\delta^H+\delta^*\right)\left(\delta^H-\delta^L\delta^*(1-p)\right)} \right) \geq 1-\delta^L+\delta^L p \tag{81}$$

The above inequality implicitly defines the exact upper bound necessary to deter low types from pooling with high types. Call this bound $\bar{\delta}^L$. Cancel $\delta^L p$ on both sides of inequality (81), multiply what is left on the right side by the denominator of the left side, take everything to the left side and write the inequality as a polynomial of δ^L, then

$$\begin{aligned}
& \delta^* \left(-\delta^H - \delta^* + \delta^H p + \delta^* p - 2\delta^H pk\right) \left(\delta^L\right)^2 \\
& + \left(\delta^H + \delta^*\right) \left(\delta^H + \delta^* - \delta^* p + \delta^H pk\right) \delta^L - \delta^H \left(\delta^H + \delta^*\right) \geq 0 \\
\iff \quad & -\frac{\delta^*\left(\delta^H+\delta^*-\delta^H p-\delta^* p+2\delta^H pk\right)}{\delta^H\left(\delta^H+\delta^*\right)} \left(\delta^L\right)^2 + \frac{\left(\delta^H+\delta^*\right)\left(\delta^H+\delta^*-\delta^* p+\delta^H pk\right)}{\delta^H\left(\delta^H+\delta^*\right)} \delta^L - 1 \geq 0 \\
\iff \quad & \overbrace{-\frac{\alpha(1+\alpha(1-p)+p(2k-1))}{(1+\alpha)}}^{\equiv \hat{A}} \left(\delta^L\right)^2 + \overbrace{(1+\alpha\,(1-p)\,+pk)}^{\equiv \hat{B}} \delta^L - 1 \geq 0 \qquad (82) \\
& \qquad\qquad \text{where } \alpha=\delta^*/\delta^H
\end{aligned}$$

$$\text{Let } Q\left(\delta^L\right) := \left(\delta^L\right)^2 \hat{A} + \delta^L \hat{B} - 1 \text{ for } \hat{A} \text{ and } \hat{B} \text{ defined above.} \tag{83}$$

If the inequality (82) binds, expressions (82)-(83) define a quadratic equation for the upper bound of δ^L with the following two solutions,

$$\bar{\delta}^L = \frac{2}{\hat{B} \pm \sqrt{\hat{B}^2 + 4\hat{A}}} \tag{84}$$

where $\hat{A}$ and $\hat{B}$ refer to expressions from equation (82). First we need to verify that $\hat{B}^2+4\hat{A}>0$ so that our solutions are real numbers

$$\begin{aligned}
\hat{B}^2 + 4\hat{A} \quad &= \quad (1+\alpha\,(1-p)+pk) - 4\frac{\alpha(1+\alpha(1-p)+p(2k-1))}{1+\alpha} > 0 \\
&= \quad \tfrac{1}{1+\alpha}(1-\alpha-\alpha^2+\alpha^3+2\alpha p-2\alpha^3 p+2kp-4\alpha kp) \\
& \quad\quad +\alpha^2 kp+\alpha^2 p^2+\alpha^3 p^2-2\alpha kp^2-2\alpha^2 kp^2+k^2p^2+\alpha k^2p^2) > 0 \\
&= \quad \tfrac{1}{1+\alpha}\left((1-\alpha)^2(1+\alpha)+2(1-\alpha)\,(\alpha(1+\alpha)\right.
\end{aligned}$$

$$+k(1-\alpha))\,p+(1+\alpha)(k-\alpha)^2p^2\big)>0,$$

the above expression is positive since every term in it is clearly positive. In addition, $\hat{A}<0,\hat{B}>0,$ so the denominator, and hence both roots defined by (84), are also real and positive. Since $\hat{A}<0,$ we know the quadratic equation $Q\left(\delta^L\right)$ defined by (83) is strictly concave with two positive, real roots. Since we are interested in the least upper bound for δ^L, the appropriate solution to $Q\left(\delta^L\right)=0$, or alternatively, the left side of ICC_{hssgl}^{L} (the deviation), exceeds the right side after if δ^L is greater than

$$\bar{\delta}^L=\frac{2}{\hat{B}+\sqrt{\hat{B}^2+4\hat{A}}}. \tag{85}$$

where $\hat{A}$ and $\hat{B}$ are defiend in expression 82. ■

Proof. (Proposition 13: DFFM equilibria) Consider the agent designated to do the first favor of size z. Let u_z^H and u_{-z}^H denote the expected payoffs, respectively, for a high type who did and did not do the initial z favor. Similarly, for a low type let u_z^L and $u_{-z}^L=p$ denote the analogous payoffs. The latter payoff is equal to p, the autarky payoff, since we are considering just one shot deviations and if the low type does not deviate in the given period by mimicking the high type then she just falls back to the specified equilibrium autarky strategy. Then for the specified equilibrium to work the incentive compatibility constraints for the designated high and low types, respectively, require that

$$\begin{aligned} ICC_D^H &: \quad \left(1-\delta^H\right)(1-z)+\delta^H u_z^H \geq 1-\delta^H+\delta^H u_{-z}^H && (86)\\ ICC_D^H &: \quad \left(1-\delta^L\right)(1-z)+\delta^L u_z^L \leq 1-\delta^L+\delta^L p && (87)\end{aligned}$$

and the continuation payoffs u_z^H, u_{-z}^H and u_z^L can be expanded into the component when facing a low type and the component when facing a high type. Namely,

$$\begin{aligned} u_z^H &= (1-\mu_o)\,p+\mu_o\bar{u}_{em}, && (88)\\ u_{-z}^H &= (1-\mu_o)\,u_{-z}^{HL}+\mu_o u_{-z}^{HH}, && (89)\\ u_z^L &= (1-\mu_o)\,p+\mu_o u_z^{LH}. && (90)\end{aligned}$$

As before, $\bar{u}_{em}$ denotes the equality matching payoff expected by an advantaged high type facing another high type. So for example, if a high type does a favor of size z this period, then her continuation payoff is just the autarky payoff p with probability $(1-\mu_o)$ since that is the likelihood he is facing a low type that will not return any favors, and with probability μ_o he is facing another high type agent so her continuation value is $\bar{u}_{em}$. Payoffs u_{-z}^{HL} and u_{-z}^{H} denote the continuation values for a high type agent who didn't do the favor this turn (i.e. deviated from the proposed equilibrium strategy), but will at the next available opportunity when facing a low type agent and a high type agent, respectively, since we are just considering one-shot deviations. Finally, u_z^{LH} denotes the expected payoff for a low type who mimicked a high type by doing a favor of size z; namely the expected value of a one-time full favor from the high type agent at the next available opportunity combined with the agent's own favor opportunities for the rest of the game. In other words

$$u_{-z}^{HL} = p\left(\left(1-\delta^H\right)(1-z)+\delta^H p\right)+(1-p)\,\delta^H u_{-z}^{HL}$$

$$
\begin{aligned}
&= p\tfrac{\left(1-\delta^H\right)(1-z)+\delta^H p}{1-\delta^H(1-p)}, && (91)\\
u_{-z}^{HH} &= p\left(\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em}\right)+(1-p)\,\delta^H u_{-z}^{HH} &&\\
&= p\tfrac{\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em}}{1-\delta^H(1-p)}, && (92)\\
u_z^{LH} &= p\left(\left(1-\delta^L\right)k+\delta^L p\right)+p\left(1-\delta^L+\delta^L u_z^{LH}\right)+(1-2p)\delta^L u_z^{LH} &&\\
&= p\tfrac{\left(1-\delta^L\right)(k+1)+\delta^L p}{1-\delta^L(1-p)}. && (93)
\end{aligned}
$$

To find the lower bound for z, substitute the expression for u_z^L from (90) and substitute it into the incentive compatibility constraint for the low type given by (87), which yields

$$\left(1-\delta^L\right)(1-z)+\delta^L\left(\left(1-\mu_o\right)p+\mu_o u_z^{LH}\right)\leq 1-\delta^L+\delta^L p.$$

We use (93) to replace u_z^{LH},

$$\left(1-\delta^L\right)(1-z)+\delta^L\left(\left(1-\mu_o\right)p+\mu_o\left(p\tfrac{\left(1-\delta^L\right)(k+1)+\delta^L p}{1-\delta^L(1-p)}\right)\right)\leq 1-\delta^L+\delta^L p.$$

And solve for z and simplify to obtain the following lower bound,

$$z\geq\tfrac{\mu_o\delta^L pk}{1-\delta^L(1-p)}\equiv\underline{z}. \tag{94}$$

At first it might look like the above lower bound could violate feasibility constraints for k large enough, but condition (5), $\delta^*>\delta^L=\frac{1}{1+p(k-1)}$, rules this out, which will be shown later, since as k grows large, the upper bound for δ^L grows proportionally small. So we need $z\geq\underline{z}$ to deter the low types from mimicking the high types. Next we need to solve for the highest z a designated high type agent would be willing to do a favor in order to signal her type and become the advantaged agent in an equality matching game if the other agent also turns out to be a high type. To this end we need to substitute the expressions for u_z^H and u_{-z}^H from (88) and (89), respectively, into the incentive compatibility constraint for the high type (86) which yields

$$\left(1-\delta^H\right)(1-z)+\delta^H\left(\left(1-\mu_o\right)p+\mu_o\bar{u}_{em}\right)\geq 1-\delta^H+\delta^H\left(\left(1-\mu_o\right)u_{-z}^{HL}+\mu_o u_{-z}^{HH}\right)$$

Next substitute in expressions for u_{-z}^{HL} and u_{-z}^{HH} from (91) and (92), respectively,

$$
\begin{aligned}
&\left(1-\delta^H\right)(1-z)+\delta^H\left(\left(1-\mu_o\right)p+\mu_o\bar{u}_{em}\right)\geq 1-\delta^H\\
&+\delta^H\left(\left(1-\mu_o\right)p\tfrac{\left(1-\delta^H\right)(1-z)+\delta^H p}{1-\delta^H(1-p)}+\mu_o p\tfrac{\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em}}{1-\delta^H(1-p)}\right)
\end{aligned}
$$

And also substitute in for $\bar{u}_{em}$ from (4) to get

$$
\begin{aligned}
&\left(1-\delta^H\right)(1-z)+\delta^H\left(\left(1-\mu_o\right)p+\mu_o\tfrac{p\left(1-\delta^H(1-p)\right)(1+k)}{1-\delta^H(1-2p)}\right)\geq 1-\delta^H\\
&+\delta^H\left(\left(1-\mu_o\right)p\tfrac{\left(1-\delta^H\right)(1-z)+\delta^H p}{1-\delta^H(1-p)}+\mu_o p\tfrac{\left(1-\delta^H\right)(1-z)+\delta^H\frac{p\left(1-\delta^H(1-p)\right)(1+k)}{1-\delta^H(1-2p)}}{1-\delta^H(1-p)}\right)
\end{aligned}
$$

Finally solve for the z and simplify, which results in

$$z \leq \min\left\{1, \frac{\mu_o \delta^H p\left(\left(1-\delta^H\right)k+\delta^H p(k-1)\right)}{\left(1-\delta^H\right)\left(1-\delta^H(1-2p)\right)}\right\} \equiv \bar{z}. \tag{95}$$

The last step is to verify that $\bar{z} \geq \underline{z}$. To this end, we show that $\underline{z}$ from (94) is increasing in δ^L :

$$\begin{aligned} \frac{\partial \underline{z}}{\partial \delta^L} &= \frac{\mu_o pk}{\left(1-\delta^L(1-p)\right)^2} > 0 \\ \implies \underline{z} \leq \underline{z}|_{\delta^L=\delta^*} &= \left.\frac{\mu_o \delta^L pk}{1-\delta^L(1-p)}\right|_{\delta^L=\frac{1}{1+p(k-1)}} = \mu_o. \end{aligned}$$

So it is enough to verify that $\bar{z} \geq \mu_o$ and incidentally this also proved that $\underline{z} \leq 1$.To show that $\bar{z} \geq \mu_o$ proceed as before:

$$\begin{aligned} \frac{\partial \bar{z}}{\partial \delta^H} &= \mu_o p \frac{\left(1-\delta^H\right)^2 k+2\delta^H p\left(1-\delta^H(1-p)\right)(k-1)}{\left(1-\delta^H\right)^2\left(1-\delta^H(1-2p)\right)^2} > 0 \\ \implies \bar{z} \geq \bar{z}|_{\delta^L=\delta^*} &= \left.\mu_o \delta^H p \frac{\left(1-\delta^H\right)k+\delta^H p(k-1)}{\left(1-\delta^H\right)\left(1-\delta^H(1-2p)\right)}\right|_{\delta^L=\frac{1}{1+p(k-1)}} = \mu_o. \end{aligned}$$

In other words, $\bar{z} \geq \underline{z}$, so this equilibrium is incentive compatible for both types for any pre-agreed $z \in [\underline{z}, \bar{z}]$ as specified by equations (94) and (95). ■

Proof. (Lemma 17: First period separating equilibria) First, observe that the belief system simply states each agent begins the game assuming that the other is a high type with probability μ_o, and if the other agent makes an observable deviation from the equilibrium path ($h_{t-1} \notin \mathcal{H}^*_{t-1}$), she is believed to be a low type. If there has been no public deviation and the other agent has either done a favor of size z_1 in the first period, or size 1 as reciprocation in some other period ($h_{t-1} \in \mathcal{H}^*_{t-1}$ and $\exists n \leq t$ s.t. $\tau_1 = \left(I_1^j, 0\right)$ or $\left(0, I_1^j\right)$ for $j = a$ or b, respectively), she is believed to be a high type. Note that $I_1^j > 0$ must be z_1 in the first period) or subsequently 1 otherwise it would be off-equilibrium path. Next if the agent receives a favor opportunity this period or did so in the first period without doing a favor ($h_{t-1} \in \mathcal{H}^*_{t-1}$ and $\{w_t^i = 1$ or $w_1^i \neq \tau_1 = 0\}$), the other agent cannot do a favor or will not do a favor per σ so beliefs about her remain as they were last period. In the latter case, they would remain μ_o forever. Another possibility is that the agent received no favor opportunity and no favor during the first period $\left(h_{t-1} \in \mathcal{H}^*_{t-1}\right.$ and $\left.w_1^i = \tau_1 = 0\right)$ so both agents revert to autarky and there will be no more informative signals on the equilibrium path. Therefore, her belief about the other agent will just be updated once from μ_o to $\frac{\mu_o(1-2p)}{1-(1+\mu_o)p}$ forever. The last possibility ("otherwise") is that the agent did an initial favor and is now waiting for reciprocation, and each period she does not receive either reciprocation or a favor opportunity, her previous belief about the other agent is updated according to Bayes' rule. It is easy to see that this belief system is consistent with strategy profile σ provided that the low types do not attempt to pool with the high types, and that the high types do not choose the autarky strategy because their favor could be wasted on a low type. Therefore, it is sufficient to check that z_1 is high enough so that a low type has no incentive to mimic a high type, and that a high type would not be better off choosing the autarky strategy. Let u_z^L denote the expected continuation payoff for a low type, say agent b, who does a favor of size z_1.

$$\begin{aligned} E\left(u_z^L|\omega^a = H\right) &\equiv u = p\left(\left(1-\delta^L\right) + \delta^L u\right) + p\left(\left(1-\delta^L\right)k + \delta^L p\right) + \left(1-2p\right)\delta^L u \\ &= p\tfrac{\left(1-\delta^L\right)(1+k)+\delta^L p}{1-\delta^L(1-p)}. \end{aligned}$$

Using the above equation, it follows that

$$\begin{aligned} u_z^L &= P\left(\omega^a = L\right)p + P\left(\omega^a = H\right)E\left(u_z^L|\omega^a = H\right) \\ &= \left(1-\mu_o\right)p + \mu_o E\left(u_z^L|\omega^a = H\right) \\ &= \left(1-\mu_o\right)p + \mu_o p\tfrac{\left(1-\delta^L\right)(1+k)+\delta^L p}{1-\delta^L(1-p)}. \end{aligned} \tag{96}$$

The incentive compatibility constraint for a low type who has received a favor opportunity in the first period not to mimic the high type is

$$\begin{aligned} ICC_z^L : \left(1-\delta^L\right)\left(1-z_1\right) + \delta^L u_z^L &\le \left(1-\delta^L\right) + \delta^L p \\ \iff u_z^L - p &\le \left(1-\delta^L\right)z_1/\delta^L \\ \iff \left(1-\mu_o\right)p + \mu_o p\tfrac{\left(1-\delta^L\right)(1+k)+\delta^L p}{1-\delta^L(1-p)} - p &\le \tfrac{1-\delta^L}{\delta^L}z_1 \text{ by (96)} \\ \implies z_1 \ge \tfrac{\mu_o\delta^L pk}{1-\delta^L(1-p)} &\equiv \underline{z}_1 \end{aligned} \tag{97}$$

Observe that $\underline{z}_1 = \mu_o\delta^L pk/\left(1-\delta^L\left(1-p\right)\right) < \mu_o\delta^* pk/\left(1-\delta^*\left(1-p\right)\right) = \mu_o < 1$. Next consider a high type, say agent a, who receives a favor opportunity in period 1. Let u_z^H denote the expected continuation payoff to a high type who does a favor of size z_1. Then

$$u_z^H = \left(1-\mu_o\right)p + \mu_o\bar{u}_{em} = \left(1-\mu_o\right)p + \mu_o\tfrac{p\left(1-\delta^H(1-p)\right)(1+k)}{1-\delta^H(1-2p)} \text{ using (4)}$$

The incentive compatibility constraint for the high type who received a favor opportunity in the first period to do a favor of size z_1 is,

$$\begin{aligned} ICC_z^H \quad &: \quad \left(1-\delta^H\right)\left(1-z_1\right) + \delta^H u_z^H \ge \left(1-\delta^H\right) + \delta^H p \\ &\iff \quad \left(1-\mu_o\right)p + \mu_o\tfrac{p\left(1-\delta^H(1-p)\right)(1+k)}{1-\delta^H(1-2p)} - p \ge \tfrac{1-\delta^H}{\delta^H}z_1 \\ &\implies \quad z_1 \le \tfrac{\mu_o\delta^H\left(\left(1-\delta^H\right)k+\delta^H p(k-1)\right)}{\left(1-\delta^H\right)\left(1-\delta^H(1-2p)\right)} \\ \therefore \text{ let} \quad & \qquad \bar{z}_1^{em} \equiv \min\left\{1, \tfrac{\mu_o\delta^H\left(\left(1-\delta^H\right)k+\delta^H p(k-1)\right)}{\left(1-\delta^H\right)\left(1-\delta^H(1-2p)\right)}\right\} \end{aligned} \tag{98}$$

To finish the proof it is necessary to show that $[\underline{z}_1, \bar{z}_1] \neq \emptyset$. Recall that $\underline{z}_1 < \mu_o$, so if $\bar{z}_1 = 1$, then it is immediate that $[\underline{z}_1, \bar{z}_1] \neq \emptyset$. Therefore suppose that $\bar{z}_1 < 1$, then it is enough to show that $\bar{z}_1 - \underline{z}_1 > 0$. To this end

$$\begin{aligned} \bar{z}_1 - \underline{z}_1 &= \tfrac{\mu_o\delta^H\left(\left(1-\delta^H\right)k+\delta^H p(k-1)\right)}{\left(1-\delta^H\right)\left(1-\delta^H(1-2p)\right)} - \tfrac{\mu_o\delta^L pk}{1-\delta^L(1-p)} \\ &= \tfrac{\delta^H\left(\left(1-\delta^H\right)k+\delta^H p(k-1)\right)}{\left(1-\delta^H\right)\left(1-\delta^H(1-2p)\right)} - \tfrac{\delta^L pk}{1-\delta^L(1-p)} \end{aligned}$$

The expression above shows that the existence of a separating equilibrium in this game does not depend on μ_o, the fraction of high types. Still supposing $\bar{z}_1 < 1$, taking the derivative of $\bar{z}_1$ with respect to δ^H in equation (98) and rearranging yields

$$\frac{d\bar{z}_1}{d\delta^H} = \mu_o p \frac{\left(1-\delta^H\right)^2 k + 2\delta^H p(k-1)\left(1-\delta^H(1-p)\right)}{\left(1-\delta^H\right)^2\left(1-\delta^H(1-2p)\right)^2}$$

From the above expression it is easy to see that $d\bar{z}_1/d\delta^H > 0$ since $k-1>0$. Therefore $\bar{z}_1$ is minimized at $\delta^H = \delta^* = 1/\left(1+p\left(k-1\right)\right)$. Substituting $1/\left(1+p\left(k-1\right)\right)$ for δ^H in the expression for $\bar{z}_1$ and simplifying yields $\bar{z}_1 = \mu_o$. Therefore $[\underline{z}_1, \bar{z}_1] \neq \emptyset$. (Again recall that $\underline{z}_1 < \mu_o$). ■

Proof. (Proposition 19: Separation between high types time restricted by mutually exclusive favor opportunity distribution) Suppose to the contrary of proposition 19 that for each period there exists $[\underline{z}_t, \bar{z}_t] \neq \emptyset$, $\underline{z}_t > 0$ of potential first favors, and an equilibrium strategy profile (σ, μ), such that if an agent does a favor of size z_t in period t, and that is the first favor in the game, then the other agent believes she is facing a high type with probability one. Furthermore, suppose (σ, μ) subsequently implements an EM game between high types at level $m\left(z_t\right)$, for some function $m : [0,1] \rightarrow [0,1]$, and that the initial favor maker (presumed high type) is the first advantaged agent in the EM game. And suppose (σ, μ) specifies the autarky strategy for low types.

To prove that (σ, μ) or any equivalent profile cannot be an equilibrium, suppose, without loss of generality, that agent b, is a low type. Then if b deviates from the autarky strategy, and does a favor of size z_t, she would receive a continuation payoff of p if she is facing another low type, and $v_{z_t}^{LH}$ if she is facing a high type, where

$$\begin{aligned} v_{z_t}^{LH} &= p\left(\left(1-\delta^L\right)\left(1+k\,m\left(z_t\right)\right)+\delta^L p\right)+\left(1-p\right)\delta^L v_{z_t}^{LH} \\ &= p\frac{\left(1-\delta^L\right)\left(1+k\;m(z_t)\right)+\delta^L p}{1-\delta^L(1-p)} = p + \frac{p\left(1-\delta^L\right)k}{1-\delta^L(1-p)} m\left(z_t\right), \end{aligned} \tag{99}$$

so her expected continuation payoff would be

$$\begin{aligned} v_{z_t}^{L} &= \left(1-\mu_t^b\right)p+\mu_t^b v_{z_t}^{LH} \\ &= \left(1-\mu_t^b\right)p+\mu_t^b p+\mu_t^b\frac{p\left(1-\delta^L\right)k}{1-\delta^L(1-p)}m\left(z_t\right) \\ &= p+\mu_t^b\frac{p\left(1-\delta^L\right)k}{1-\delta^L(1-p)}m\left(z_t\right) \end{aligned} \tag{100}$$

And if agent b does not do the favor, her continuation payoff is p if agent a is a low type, and $v_{-z_t}^{LH}$ if a is a high type, where

$$\begin{aligned} v_{-z_t}^{LH} &= p\left(1-\delta^L+\delta^L v_{-z_{t+1}}^{LH}\right)+p\left(\left(1-\delta^L\right)kz_{t+1}+\delta^L p\right)+\left(1-2p\right)\delta^L v_{-z_{t+1}}^{LH} \\ &= p\left(1-\left(1-p\right)\delta^L\right)+\underbrace{p\left(1-\delta^L\right)k}_{\equiv b}z_{t+1}+\underbrace{\left(1-p\right)\delta^L}_{\equiv a}v_{-z_{t+1}}^{LH} \\ &= p\left(1-a\right)+bz_{t+1}+a\left(p\left(1-a\right)+bz_{t+2}+a\left(v_{-z_{t+2}}^{LH}\right)\right), \end{aligned}$$

iteratively expanding each $v_{\cdot}^{LH}$ and collecting like terms
produces the following geometric sums

$$
\begin{aligned}
&= p\left(1-a\right)\left(1+a+a^2+...\right)+b\left(z_{t+1}+az_{t+2}+a^2z_{t+3}+...\right)\\
&= p+b\sum_{i=0}^{\infty}a^iz_{t+1+i}=p+p\left(1-\delta^L\right)kS^{LH},\\
&\quad \text{where } S^{LH}=\sum_{i=0}^{\infty}\left(\left(1-p\right)\delta^L\right)^iz_{t+1+i}.
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
v_{-z_t}^L &= \left(1-\mu_t^b\right)p+\mu_t^b\left[p+p\left(1-\delta^L\right)kS^{LH}\right]\\
&= p+\mu_t^bp\left(1-\delta^L\right)kS^{LH} \qquad (101)
\end{aligned}
$$

In order for agent b not to pool with high types we need z_t large enough that her incentive compatibility constraint to follow the autarky strategy instead of deviating is satisfied for all t.

$$
\begin{aligned}
ICC_z^L:\left(1-\delta^L\right)\left(1-z_t\right)+\delta^Lv_{z_t}^L \leq &\quad \left(1-\delta^L\right)+\delta^Lv_{-z_t}^L\\
\iff z_t \geq &\quad \tfrac{\delta^L}{1-\delta^L}\left(v_{z_t}^L-v_{-z_t}^L\right)
\end{aligned}
$$

Substituting in for $v_{z_t}^L$ and $v_{-z_t}^L$ from (100) and (101)yields,

$$
\begin{aligned}
z_t &\geq \tfrac{\delta^L}{\left(1-\delta^L\right)}\left(p+\mu_t^b\tfrac{p\left(1-\delta^L\right)k}{1-\delta^L(1-p)}m\left(z_t\right)-p-\mu_t^bp\left(1-\delta^L\right)kS^{LH}\right)\\
&= \mu_t^b\tfrac{\delta^Lp\left(1-\delta^L\right)k}{1-\delta^L}\left(\tfrac{m(z_t)}{1-\delta^L(1-p)}-S^{LH}\right)\\
&= \mu_t^b\delta^Lpk\left(\tfrac{m(z_t)}{1-\delta^L(1-p)}-S^{LH}\right),\\
&= \tfrac{\mu_t^b\delta^Lpk}{1-\delta^L(1-p)}\left(m\left(z_t\right)-z_t^{LH}\right),\ \text{where } z_t^{LH}\in\left(0,\bar{z}_{t+1}\right)\ \text{is s.t.}\\
&\quad \sum_{i=0}^{\infty}\left(\left(1-p\right)\delta^L\right)^iz_{t+1+i}=\sum_{i=0}^{\infty}\left(\left(1-p\right)\delta^L\right)^iz_t^{LH}=\tfrac{z_t^{LH}}{1-\delta^L(1-p)}. \qquad (102)
\end{aligned}
$$

The incentive compatibility constraint needs to hold for every t and every possible history, so consider the history that maximizes the right-hand side, $h_t^b=\{1,1,...,1\}$, and a public history of no favors assuming agent b follows the autarky strategy as specified by σ for low types. Then $\mu_t^b=\mu_o$ and the right-hand side of the low type's incentive compatibility constraint satisfies

$$
\begin{aligned}
\tfrac{\mu_t^b\delta^Lpk}{1-\delta^L(1-p)}\left(m\left(z_t\right)-z_t^{LH}\right) &\leq \tfrac{\mu_o\delta^Lpk}{1-\delta^L(1-p)}\left(m\left(z_t\right)-z_t^{LH}\right)\\
&= \frac{m\left(z_t\right)-z_t^{LH}}{M}\ \text{by def'n of } M \text{ from (48).}
\end{aligned}
$$

Then z_t must satisfy

$$
z_t\geq\frac{m\left(z_t\right)-z_t^{LH}}{M},\forall t, \qquad (103)
$$

or else there exists a set of histories, that occur with strictly positive probability, such that a low type would could profit by mimicking a high type. Conversely, if (103) holds, the low types will never pool with high types, and given a sequence $z\equiv\{z_t\}_{t=1}^{\infty}$, condition (103) represents the greatest lower bound on the individual terms of the sequence z. Rearranging we obtain condition

$$m\left(z_t\right) \leq M z_t + z_t^{LH}, \forall t, \tag{104}$$

for the least upper bound on $m\left(z_t\right)$ conditional on sequence z specified by (σ, μ). And independent of z,

$$\begin{aligned} & m\left(z_t\right) < M z_t + \bar{z}_{t+1} \text{ since } z_t^{LH} \in \left(0, \bar{z}_{t+1}\right) \\ \Longrightarrow \quad & m\left(z_t\right) < M z_t + 1, \forall z_t \text{ and } m\left(\bar{z}_t\right) < (M+1)\, \bar{z}_t \end{aligned} \tag{105}$$

since $\bar{z}_t$ is decreasing in t. Furthermore, if z is a decreasing sequence, (104) implies that

$$\begin{aligned} m\left(z_t\right) & \leq M z_t + z_t \\ \Longrightarrow \frac{m\left(z_t\right)}{z_t} & \leq M+1 \end{aligned} \tag{106}$$

Observe that the above condition rules out functions of the form $m\left(z_t\right) = c \in (0,1]$ (constant bounds), for example. Given a sequence of z_t, let $t^* = \inf\left\{t \in \mathbb{N} : z_{t^*} \leq \frac{c}{M+1}\right\}$ and suppose $h_{t^*}^b = \{1, 1, ..., 1\}$, then $\mu_{t^*}^b = \mu_o$ so the low type's incentive compatibility constraint to follow the autarky strategy would be violated. In this example, the violation would occur with probability greater than p^{t^*}. Also, note that $t^* \neq \infty$ since $z_t \rightarrow 0$ as $t \rightarrow \infty$ because a high type grows more pessimistic each period that goes by without an initial favor, and therefore the largest favor she is willing to do to separate must also be decreasing.[4] If $m\left(z_t\right)$ is chosen appropriately, for example $m\left(z_t\right) = \min\left\{(M+1)\, z_t, 1\right\}$, then low types would have no incentive to mimic high types even if they knew they were facing one with certainty. However, we still need to consider the incentive compatibility constraints for the high type, say, without loss of generality, for agent a

$$ICC_z^H : \left(1-\delta^H\right)\left(1-z_t\right) + \delta^H u_{z_t}^H \geq \left(1-\delta^H\right) + \delta^H u_{-z_t}^H,$$

where $u_{z_t}^H$ and $u_{-z_t}^H$ are the expected continuation payoffs for a high type if she does a favor of size z_t and if she does no favor, respectively. Then these payoffs can be broken down into two separate components, the payoffs when facing a high and a low type, respectively,

$$\begin{aligned} u_{z_t}^H & = \left(1-\mu_t^a\right) u_{z_t}^{HL} + \mu_t^a u_{z_t}^{HH}, \text{ and} \\ u_{-z_t}^H & = \left(1-\mu_t^a\right) u_{-z_t}^{HL} + \mu_t^a u_{-z_t}^{HH}. \end{aligned}$$

Clearly $u_{z_t}^{HL} = p$, the autarky payoff, since σ dictates that agent a, who did the first favor, do no more favors until reciprocation is received, and that agent b, a low type in this instance, do no favors. If b is a high type, a's expected payoff is $u_{z_t}^{HH} = \bar{u}_{em(m(z_t))} = p + B\, m\left(z_t\right)$ per (71). To compute the components of $u_{-z_t}^H$ recall that we are considering just one-shot deviations, so next period agent a is assumed to do a favor of size z_{t+1} if she receives a favor opportunity, or she could receive a favor of size z_{t+1} from agent b if b is a high type, and so forth. Namely,

$$u_{-z_t}^{HL} = p\left(\left(1-\delta^H\right)\left(1-z_{t+1}\right) + \delta^H p\right) + (1-p)\, \delta^H u_{-z_{t+1}}^{HL}$$

---

[4] The initial favors under (σ, μ) that are incentive compatible for a high type could be weakly decreasing at first (sequence of full favors), but must become strictly decreasing once the high type grows sufficiently pessimistic that her incentive compatibility constraint binds.

$$
\begin{aligned}
&= p\left(1-(1-p)\,\delta^{H}\right) - \underbrace{p\left(1-\delta^{H}\right)}_{\equiv b} z_{t+1} + \underbrace{(1-p)\,\delta^{H}}_{\equiv a} u^{HL}_{-z_{t+1}} \\
&= p\,(1-a) - bz_{t+1} + a\left(p\,(1-a) - bz_{t+2} + a\left(u^{HL}_{-z_{t+2}}\right)\right), \\
&\quad \text{iteratively expanding each } v^{HL}_{\cdot} \text{ and collecting like} \\
&\quad \text{terms produces the following geometric sums} \\
&= p\,(1-a)\left(1+a+a^{2}+...\right) - b\left(z_{t+1}+az_{t+2}+a^{2}z_{t+3}+...\right) \\
&= p\tfrac{1-a}{1-a} - b\sum_{i=0}^{\infty} a^{i} z_{t+1+i} = p - p\left(1-\delta^{H}\right) S^{HL}, \\
&\quad \text{where } S^{HL} = \sum_{i=0}^{\infty}\left((1-p)\,\delta^{H}\right)^{i} z_{t+1+i}.
\end{aligned}
$$

Performing a similar calculation for $u^{HH}_{-z_t}$ yields

$$
\begin{aligned}
u^{HH}_{-z_t} &= p\left(\left(1-\delta^{H}\right)(1-z_{t+1}) + \delta^{H}\bar{u}_{em(m(z_{t+1}))}\right) \\
&\quad +p\left(\left(1-\delta^{H}\right)kz_{t+1} + \delta^{H}\underline{u}_{em(m(z_{t+1}))}\right) + (1-2p)\,\delta^{H}u^{HH}_{-z_{t+1}} \\
&= p\left(1-\delta^{H}\right) + p\delta^{H}\left(\bar{u}_{em(m(z_{t+1}))} + \underline{u}_{em(m(z_{t+1}))}\right) \\
&\quad +p\left(1-\delta^{H}\right)(k-1)\,z_{t+1} + (1-2p)\,\delta^{H}u^{HH}_{-z_{t+1}} \\
&= p\left(1-\delta^{H}\right) + p\delta^{H}\left(2p + p\,(k-1)\,m\,(z_{t+1})\right) \\
&\quad +p\left(1-\delta^{H}\right)(k-1)\,z_{t+1} + (1-2p)\,\delta^{H}u^{HH}_{-z_{t+1}} \text{ by (72)} \\
&= p\left(1-\delta^{H}\,(1-2p)\right) \\
&\quad +\underbrace{p\,(k-1)}_{\equiv b}\underbrace{\left(p\delta^{H}m\,(z_{t+1}) + \left(1-\delta^{H}\right)z_{t+1}\right)}_{\equiv n(z_{t+1})} + \underbrace{(1-2p)\,\delta^{H}}_{\equiv a}u^{HH}_{-z_{t+1}} \\
&= p\,(1-a) + bn\,(z_{t+1}) \\
&\quad +a\left(p\,(1-a) + b\,n\,(z_{t+2}) + a\,u^{H}_{-z_{t+2}}\right). \text{ Iteratively expanding each } u^{H}_{\cdot} \text{ and} \\
&\quad \text{collecting like terms produces the following geometric sums} \\
&= a\left(1+a+a^{2}+...\right) + b\left(n\,(z_{t+1}) + a\,n\,(z_{t+2}) + a^{2}n\,(z_{t+3}) + ...\right) \\
&= p\tfrac{1-a}{1-a} + b\sum_{i=0}^{\infty} a^{i}n\,(z_{t+1+i}) = p + p\,(k-1)\,S^{HH}, \text{ where} \\
&\qquad S^{HH} \quad = \textstyle\sum_{i=0}^{\infty}\left((1-2p)\,\delta^{H}\right)^{i} n\,(z_{t+1+i}) \\
&\qquad\qquad\quad\;\; = \textstyle\sum_{i=0}^{\infty}\left((1-2p)\,\delta^{H}\right)^{i}\left(p\delta^{H}m\,(z_{t+1+i}) + \left(1-\delta^{H}\right)z_{t+1+i}\right)
\end{aligned}
$$

Putting the pieces together, we have

$$
\begin{aligned}
u^{H}_{z_t} &= (1-\mu^{a}_{t})\,p + \mu^{a}_{t}\,(p + Bm\,(z_t)), \text{ and} \\
u^{H}_{-z_t} &= (1-\mu^{a}_{t})\left(p - p\left(1-\delta^{H}\right)S^{HL}\right) + \mu^{a}_{t}\left(p + p\,(k-1)\,S^{HH}\right).
\end{aligned}
$$

Now rearranging ICC^{H}_{z} and plugging in the above values for $u^{H}_{z_t}$ and $u^{H}_{-z_t}$ we have

$$
\begin{aligned}
z_t &\leq \tfrac{\delta^{H}}{1-\delta^{H}}\left(u^{H}_{z_t} - u^{H}_{-z_t}\right) \\
&= \tfrac{\delta^{H}}{1-\delta^{H}}\,(1-\mu^{a}_{t})\left(p - \left(p - p\left(1-\delta^{H}\right)S^{HL}\right)\right)
\end{aligned}
$$

$$\begin{aligned} & +\tfrac{\delta^H}{1-\delta^H}\mu_t^a\left(p+B\,m\left(z_t\right)-\left(p+p\left(k-1\right)S^{HH}\right)\right) \\ = \; & \left(1-\mu_t^a\right)p\,\delta^H S^{HL}+\mu_t^a\tfrac{\delta^H}{1-\delta^H}\left(B\,m\left(z_t\right)-p\left(k-1\right)S^{HH}\right). \end{aligned} \tag{107}$$

Because $\bar{z}_t$ is decreasing,

$$p\delta^H S^{HL}=p\delta^H\sum_{i=0}^{\infty}\left(\left(1-p\right)\delta^H\right)^i z_{t+1+i}\leq p\delta^H\sum_{i=0}^{\infty}\left(\left(1-p\right)\delta^H\right)^i\bar{z}_t=\tfrac{p\delta^H}{1-\delta^H(1-p)}\bar{z}_t.$$

And because $S^{HH}>0$,

$$z_t\leq\left(1-\mu_t^a\right)\tfrac{p\delta^H}{1-\delta^H(1-p)}\bar{z}_t+\mu_t^a\tfrac{\delta^H B}{1-\delta^H}m\left(z_t\right)\leq\left(1-\mu_t^a\right)\tfrac{p\delta^H}{1-\delta^H(1-p)}\bar{z}_t+\mu_t^a\tfrac{\delta^H B}{1-\delta^H}m\left(\bar{z}_t\right).$$

The last inequality must hold for all t and all $z_t\in\left[\underline{z}_t,\bar{z}_t\right]$, otherwise we would have a contradiction to our initial supposition, and the proof would be over. So let $z_t=\bar{z}_t$, then the following must hold

$$\begin{aligned} \bar{z}_t & \leq \left(1-\mu_t^a\right)\tfrac{p\delta^H}{1-\delta^H(1-p)}\bar{z}_t+\mu_t^a\tfrac{\delta^H B}{1-\delta^H}m\left(\bar{z}_t\right) \\ & < c\bar{z}_t+\mu_t^a d\,m\left(z_t\right) \text{ for } c\equiv\tfrac{\delta^H p}{1-\delta^H(1-p)}, d\equiv\tfrac{\delta^H B}{1-\delta^H} \\ \Longrightarrow \tfrac{1-c}{d} & < \mu_t^a\tfrac{m(z_t)}{z_t}\leq\mu_t^a M \text{ by (48).} \end{aligned} \tag{108}$$

It is easy to verify that $\frac{1-c}{d}>0$. Namely from (71),

$$\begin{aligned} & d=\tfrac{\delta^H}{1-\delta^H}B=\tfrac{\delta^H}{1-\delta^H}p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}>0 \text{ and } \delta^H<1 \\ \Longrightarrow \; & \delta^H p<1-\delta^H\left(1-p\right)\Longrightarrow c=\tfrac{\delta^H p}{1-\delta^H(1-p)}<1 \\ \therefore \; & \tfrac{1-c}{d}>0. \end{aligned}$$

Furthermore, $\frac{1-c}{d}$ and M are fixed while $\mu_t^a\rightarrow 0$ in probability as $t\rightarrow\infty$ unless separation takes place, so we know that with some positive probability condition (108) will be violated producing a contradiction to our supposition that (σ,μ) was a PBE. To compute the exact time we need the following lemma.

Lemma 19a: Let s denote the number of periods agent i has received neither a favor opportunity, nor a favor from the other agent, then

$$\mu_s^i=\frac{\mu_o\left(1-2p\right)^s}{\mu_o\left(1-2p\right)^s+\left(1-\mu_o\right)\left(1-p\right)^s}. \tag{109}$$

Proof of lemma 19a: For $s=1$, we know that

$$\mu_1^i=\frac{\mu_o\left(1-2p\right)}{1-\left(1+\mu_o\right)p}=\frac{\mu_o\left(1-2p\right)^1}{\mu_o\left(1-2p\right)^1+\left(1-\mu_o\right)\left(1-p\right)^1}$$

Suppose (109) for some $s\in\mathbb{N}$, then

$$\mu_{s+1}^i = \tfrac{\mu_s^i(1-2p)}{1-(1+\mu_s^i)p}=\tfrac{\mu_s^i(1-2p)}{1-p-p\mu_s^i} \text{ per (7 (iv))}$$

$$\begin{aligned}
&= \frac{\frac{\mu_o(1-2p)^s}{\mu_o(1-2p)^s+(1-\mu_o)(1-p)^s}(1-2p)}{1-p-p\frac{\mu_o(1-2p)^s}{\mu_o(1-2p)^s+(1-\mu_o)(1-p)^s}} \\
&= \frac{\mu_o(1-2p)^{s+1}}{(1-p)(\mu_o(1-2p)^s+(1-\mu_o)(1-p)^s)-p\mu_o(1-2p)^s} \\
&= \frac{\mu_o(1-2p)^{s+1}}{(1-2p)\mu_o(1-2p)^s+(1-p)(1-\mu_o)(1-p)^s} \\
&= \frac{\mu_o(1-2p)^{s+1}}{\mu_o(1-2p)^{s+1}+(1-\mu_o)(1-p)^{s+1}}
\end{aligned}$$

So if (109) holds for some $s \in \mathbb{N}$, it holds for $s+1$. We know it holds for $s=1$, therefore by induction it must hold for all $s \in \mathbb{N}$. Let $t^* := \inf\left\{t \in \mathbb{N} : \frac{\mu_o(1-2p)^{t-1}}{\mu_o(1-2p)^{t-1}+(1-\mu_o)(1-p)^{t-1}} < \frac{1-a}{bM}\right\}$ and suppose $h^a_{t^*} = \{0, ..., 0, 1\}$ and $H_{t^*} = \{0, ..., 0\}$, then $\mu^a_{t^*} = \frac{\mu_o(1-2p)^{t^*-1}}{\mu_o(1-2p)^{t^*-1}+(1-\mu_o)(1-p)^{t^*-1}} < \frac{1-a}{bM}$ by (109) and by definition of t^* so that the incentive compatibility constraint for the high type is violated by period t^* if she does not receive any favor opportunities or favors before then. Hence separation has to occur within t^*-1 or fewer periods or it will never take place. The belief function follows from Bayesian updating and must have this form given a history of no favor opportunities until now and no favors in order for μ to be consistent with σ. Thus separation cannot be guaranteed with probability 1 in any SS equilibria. ■

Proof. (Lemma 25: EM with independent favor opportunities) Suppose we start the game with agent b as the advantaged agent, and the level of trust is $z \in (0,1]$, that is agent a does a favor of size $x = z$ if she receives a favor opportunity, and then does no further favors until the other agent, in this case agent b, reciprocates by doing a favor of size $y = z$. It follows that in terms of average discounted payoffs

$$\begin{aligned}
\underline{u}_{em(z)} &= p\left(\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em(z)}\right)+(1-p)\,\delta^H\underline{u}_{em(z)} \\
&= p\frac{\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em(z)}}{1-\delta^H(1-p)} \\
\bar{u}_{em(z)} &= p^2\left(\left(1-\delta^H\right)(kz+1)+\delta^H\underline{u}_{em(z)}\right)+p\,(1-p)\left(\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em(z)}\right) \\
&\quad +(1-p)\,p\left(\left(1-\delta^H\right)kz+\delta^H\underline{u}_{em(z)}\right)+(1-p)^2\,\delta^H\bar{u}_{em(z)} \\
&= p^2\left(\left(1-\delta^H\right)(kz+1)+\delta^H\underline{u}_{em(z)}\right)+p\left(\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em(z)}\right) \\
&\quad -p^2\left(\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em(z)}\right)+p\left(\left(1-\delta^H\right)kz+\delta^H\underline{u}_{em(z)}\right) \\
&\quad -p^2\left(\left(1-\delta^H\right)kz+\delta^H\underline{u}_{em(z)}\right)+(1-2p)\,\delta^H\bar{u}_{em(z)}+p^2\delta^H\bar{u}_{em(z)} \\
&= p\left(\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em(z)}\right)+p\left(\left(1-\delta^H\right)kz+\delta^H\underline{u}_{em(z)}\right) \\
&\quad +(1-2p)\,\delta^H\bar{u}_{em(z)} \\
&= p\frac{\left(1-\delta^H\right)(1+kz)+\delta^H\underline{u}_{em(z)}}{1-\delta^H(1-p)}.
\end{aligned}$$

Equations for $\underline{u}_{em(z)}$ and $\bar{u}_{em(z)}$ are identical to their counterparts from the mutually exclusive favor opportunities case (see proof of lemma 2.1), and the incentive compatibility constraint is also the same, so by symmetry:

$$\underline{u}_{em(z)} = p + Az \text{ where } A \equiv p\frac{\delta^H p(k-1)-\left(1-\delta^H\right)}{1-\delta^H(1-2p)},$$

$$\bar{u}_{em(z)} \;=\; p + Bz \text{ where } B \equiv p\tfrac{\delta^H p(k-1)+(1-\delta^H)k}{1-\delta^H(1-2p)}$$

and $\delta^H \geq \delta^* = \frac{1}{1+p(k-1)}$ is necessary and sufficient to implement any simple EM strategy profile, and setting $x, y = 1$ ensures the greatest gains from cooperation. ■

Proof. (Lemma 26: EM without initial designations) Once agent designations are determined, the game will be exactly as before in 25. In particular, the payoffs and incentive compatibility constraints are the same as before. Therefore we can focus on the initial stage of the game before designations are determined. Let $\hat{u}^z_{em}$ denote the expected payoff for an agent with no designation. Then,

$$\begin{aligned}
\hat{u}^z_{em} &= p^2\left(\left(1-\delta^H\right)\left(1+\left(k-1\right)z\right)+\delta^H\hat{u}^z_{em}\right) \\
&\quad +p\left(1-p\right)\left(\left(1-\delta^H\right)\left(1-z\right)+\delta^H\bar{u}_{em}\right) \\
&\quad +\left(1-p\right)\left(p\left(\left(1-\delta^H\right)kz+\delta^H\underline{u}_{em}\right)+\left(1-p\right)\delta^H\hat{u}^z_{em}\right) \\
&= \tfrac{(1-\delta^H)p(1-z+kz)+\delta^H(p-p^2)(\underline{u}_{em}+\bar{u}_{em})}{1-\delta^H+2\delta^H p-2\delta^H p^2} \\
&= \tfrac{(1-\delta^H)p(1-z+kz)+\delta^H(p-p^2)p(k-1)}{1-\delta^H+2\delta^H p-2\delta^H p^2} \\
&= p+\underbrace{\tfrac{\delta^H(1-p)p^2(k-1)}{1-\delta^H+2\delta^H p(1-p)}}_{\equiv C_1}+\underbrace{\tfrac{(1-\delta^H)p(k-1)}{1-\delta^H+2\delta^H p(1-p)}}_{\equiv C_2}z. \qquad (110)
\end{aligned}$$

The incentive compatibility constraint for an undesignated agent to do a favor of size z is

$$\begin{aligned}
ICC^{nd}_{em} \quad &: \quad p\left(\left(1-\delta^H\right)\left(1+\left(k-1\right)z\right)+\delta^H\hat{u}^z_{em}\right)+\left(1-p\right)\left(\left(1-\delta^H\right)\left(1-z\right)+\delta^H\bar{u}_{em}\right) \\
&\quad \geq p\left(\left(1-\delta^H\right)\left(1+kz\right)+\delta^H\underline{u}_{em}\right)+\left(1-p\right)\left(\left(1-\delta^H\right)+\delta^H\hat{u}^z_{em}\right) \\
&\iff p\left(\left(1-\delta^H\right)\left(-z\right)+\delta^H\left(\hat{u}^z_{em}-\underline{u}_{em}\right)\right) \\
&\quad +\left(1-p\right)\left(\left(1-\delta^H\right)\left(-z\right)+\delta^H\left(\bar{u}_{em}-\hat{u}^z_{em}\right)\right)\geq 0 \\
&\iff p\delta^H\left(\hat{u}^z_{em}-\underline{u}_{em}\right)+\left(1-p\right)\delta^H\left(\bar{u}_{em}-\hat{u}^z_{em}\right)-\left(1-\delta^H\right)z\geq 0 \\
&\iff p\delta^H\left(\hat{u}^z_{em}-\underline{u}_{em}\right)+\left(1-p\right)\delta^H\left(\bar{u}_{em}-\hat{u}^z_{em}\right)-\left(1-\delta^H\right)z\geq 0
\end{aligned}$$

substituting in for $\hat{u}^z_{em}, \underline{u}_{em}$ and $\bar{u}_{em}$ from (110), (69) and (71) yields

$$\begin{aligned}
&p\delta^H\left(p+C_1+C_2z-p-A\right)+\left(1-p\right)\delta^H\left(p+B-p-C_1-C_2z\right)-\left(1-\delta^H\right)z\geq 0 \\
&\qquad \iff p\delta^H\left(C_1+C_2z-A\right)+\left(1-p\right)\delta^H\left(B-C_1-C_2z\right)-\left(1-\delta^H\right)z\geq 0
\end{aligned}$$

Solving for δ^H from the above constraint is complicated, but we can we solve for z and show that $z \leq 1/2$, is required when $\delta^H = \delta^*$. In particular, solving for z from the previous inequality yields

$$z \leq \tfrac{\delta^H p\left(p(1-\delta^H(1-p))+k\left(1-\delta^H-p+3\delta^H p-3\delta^H p^2\right)\right)}{(1-\delta^H(1-2p))\left(1-\delta^H(1-(1+k)p+2kp^2)\right)} \leq 1/2 \text{ for } \delta^H = \delta^*.$$

It follows immediately that the upper bound on z is increasing with δ^H since the more patient agents are the higher cost they will pay for tomorrow's continuation promise. ■

Proof. (Proposition 28: SS equilibria with independent favor opportunities) The incentive compatibility constraint for a high type to do a favor of size z_t in period t if $H_{t-1} = \{0, 0, ..., 0\}$

is provided below.

$$\begin{aligned}
ICC^H_{iss} \quad : \quad & \left(1-\delta^H\right)\left(1-z_t\right)+\left(1-\mu_t\right)\delta^H p \\
& +\mu_t\left(p\left(\left(1-\delta^H\right)kz_t+\delta^H\hat{u}_{em}\right)+\left(1-p\right)\delta^H\bar{u}_{em}\right) \\
& \geq 1-\delta^H+\left(1-\mu_t\right)\delta^H\hat{u}^{HL}_{-z_t} \\
& +\mu_t\left(p\left(\left(1-\delta^H\right)kz_t+\delta^H\underline{u}_{em}\right)+\left(1-p\right)\delta^H\hat{u}^{H}_{-z_t}\right) \\
\Longrightarrow \quad & \tfrac{\delta^H}{1-\delta^H}\left(\mu_t p\left(\hat{u}_{em}-\underline{u}_{em}\right)\right. \\
& \left.+\,\mu_t\left(1-p\right)\left(\bar{u}_{em}-\hat{u}^{HH}_{-z_t}\right)+\left(1-\mu_t\right)\left(p-\hat{u}^{HL}_{-z_t}\right)\right)\geq z_t,
\end{aligned}\tag{111}$$

where $\mu^i_t=\mu_t$ for both agents regardless of type since neither one has done a favor yet and private favor opportunities are uninformative about the type of the other agent since favor opportunities are independent. Payoffs $\hat{u}^{HH}_{-z_t}$ and $\hat{u}^{HL}_{-z_t}$ denote the continuation values implemented by (σ,μ) for a high type agent who did not do or receive a favor in period t facing a high type and a low type, respectively. As for the low type, the following incentive compatibility constraint has to be satisfied for her not to mimic a high type.

$$\begin{aligned}
ICC^L_{iss} \quad : \quad & \left(1-\delta^L\right)\left(1-z_t\right) \\
& +\mu_t\left(p\left(\left(1-\delta^L\right)kz_t+\delta^L\bar{u}^L_{em}\right)+\left(1-p\right)\delta^L\bar{u}^L_{em}\right)+\left(1-\mu_t\right)\delta^L p \\
\leq \quad & \left(1-\delta^L\right)+\mu_t\left(p\left(\left(1-\delta^L\right)kz_t+\delta^L p\right)+\left(1-p\right)\delta^L\hat{u}^{LH}_{-z_t}\right)+\left(1-\mu_t\right)\delta^L p \\
\Longrightarrow \quad & \tfrac{\delta^L}{1-\delta^L}\mu_t\left(p\left(\bar{u}^L_{em}-p\right)+\left(1-p\right)\left(\bar{u}^L_{em}-\hat{u}^{LH}_{-z_t}\right)\right)\leq z_t,
\end{aligned}\tag{112}$$

where $\bar{u}^L_{em}$ denotes the expected payoff to an advantaged low type facing a high type in an EM game of full trust, and $\hat{u}^{LH}_{-z_t}$ denotes the continuation payoff a low type if she is facing a high type in period t. That is, her autarky payoff and one time small favor as soon as the high type receives a favor opportunity. Per inequalities (111) and (112), it is necessary to prove that

$$\begin{aligned}
& \tfrac{\mu_t\delta^H}{1-\delta^H}\left(p\left(\hat{u}_{em}-\underline{u}_{em}\right)+\left(1-p\right)\left(\bar{u}_{em}-\hat{u}^{HH}_{-z_t}\right)\right)+\tfrac{\left(1-\mu_t\right)\delta^H}{1-\delta^H}\left(p-\hat{u}^{HL}_{-z_t}\right) \\
& \qquad \geq \tfrac{\mu_t\delta^L}{1-\delta^L}\left(p\left(\bar{u}^L_{em}-p\right)+\left(1-p\right)\left(\bar{u}^L_{em}-\hat{u}^{LH}_{-z_t}\right)\right),\forall t
\end{aligned}\tag{113}$$

and for $z_t\in(0,1]$.First, we need to solve for $\hat{u}^{HH}_{-z_t}$, $\hat{u}^{HL}_{-z_t}$, $\bar{u}^L_{em}$ and $\hat{u}^{LH}_{-z_t}$.

$$\begin{aligned}
\bar{u}^L_{em} &= p\left(1-\delta^L\right)+p\left(\left(1-\delta^L\right)\tfrac{1}{2}k+\delta^L p\right)+\left(1-p\right)\delta^L\bar{u}^L_{em} \\
&= p+\tfrac{\left(1-\delta^L\right)pk}{1-\delta^L\left(1-p\right)}.
\end{aligned}\tag{114}$$

Repeating the calculation for $\hat{u}^{LH}_{-z_t}$:

$$\begin{aligned}
\hat{u}^{LH}_{-z_t} &= p\left(1-\delta^L\right)+p\left(\left(1-\delta^L\right)kz_{t+1}+\delta^L p\right)+\delta^L\left(1-p\right)\hat{u}^{LH}_{-z_{t+1}} \\
&= \underbrace{p\left(1-\delta^L\left(1-p\right)\right)}_{\equiv a_1}+\underbrace{p\left(1-\delta^L\right)k}_{\equiv a_2}z_{t+1}+\underbrace{\delta^L\left(1-p\right)}_{\equiv d_L}\hat{u}^{LH}_{-z_{t+1}} \\
&= a_1+a_2z_{t+1}+d_L\left(a_1+a_2z_{t+2}+d_L\hat{u}^{LH}_{-z_{t+2}}\right) \\
&= a_1\left(1+d_L+d_L^2+...\right)+a_2\left(z_{t+1}+d_Lz_{t+2}+d_L^2z_{t+3}+...\right)
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{a_1}{1-d_L} + a_2 \sum_{i=0}^{\infty} d_L^i z_{t+1+i} = \tfrac{p\left(1-\delta^L(1-p)\right)}{1-\delta^L(1-p)} + a_2 \sum_{i=0}^{\infty} d_L^i z_{t+1+i} \\
&= p + \left(1-\delta^L\right) S_{iLH} \text{ for } S_{iLH} \equiv pk \sum_{i=0}^{\infty} \left(\delta^L\left(1-p\right)\right)^i z_{t+1+i}. \qquad (115)
\end{aligned}
$$

The math remains the same for $\hat{u}^{HL}_{-z_t}$:

$$
\begin{aligned}
\hat{u}^{HL}_{-z_t} &= p\left(\left(1-\delta^H\right)\left(1-z_{t+1}\right) + \delta^H p\right) + \delta^H \left(1-p\right) \hat{u}^{HL}_{-z_{t+1}} \\
&= \underbrace{p\left(1-\delta^H\left(1-p\right)\right)}_{\equiv b_1} - \underbrace{p\left(1-\delta^H\right)}_{\equiv b_2} z_{t+1} + \underbrace{\delta^H\left(1-p\right)}_{\equiv d_{hl}} \hat{u}^{HL}_{-z_{t+1}} \\
&= b_1 - b_2 z_{t+1} + d_{hl}\left(b_1 - b_2 z_{t+2} + d_{hl} u^{HL}_{-z_{t+2}}\right) \\
&\quad + b_1\left(1 + d_{hl} + d_{hl}^2 + \ldots\right) - b_2\left(z_{t+1} + d_{hl} z_{t+2} + d_{hl}^2 z_{t+3} + \ldots\right) \\
&= \tfrac{b_1}{1-d_{hl}} - b_2 \sum_{i=0}^{\infty} d_{hl}^i z_{t+1+i} = \tfrac{p\left(1-\delta^H(1-p)\right)}{1-\delta^H(1-p)} - b_2 \sum_{i=0}^{\infty} d_{hl}^i z_{t+1+i} \\
&= p - \left(1-\delta^H\right) S_{iHL} \text{ for } S_{iHL} \equiv p \sum_{i=0}^{\infty} \left(\delta^H\left(1-p\right)\right)^i z_{t+1+i}. \qquad (116)
\end{aligned}
$$

And finally for $\hat{u}^{HH}_{-z_t}$:

$$
\begin{aligned}
\hat{u}^{HH}_{-z_t} &= p^2\left(\left(1-\delta^H\right)\left(1+\left(k-1\right)z_{t+1}\right) + \delta^H \hat{u}_{em}\right) \\
&\quad + p\left(1-p\right)\left(\left(1-\delta^H\right)\left(1-z_{t+1}\right) + \delta^H \bar{u}_{em}\right) \\
&\quad + \left(1-p\right) p\left(\left(1-\delta^H\right) k z_{t+1} + \delta^H \underline{u}_{em}\right) + \left(1-p\right)^2 \delta^H \hat{u}^{HH}_{-z_t} \\
&= p^2\left(\left(1-\delta^H\right) + \delta^H \hat{u}_{em}\right) + p\left(1-p\right)\left(\left(1-\delta^H\right) + \delta^H\left(\bar{u}_{em} + \underline{u}_{em}\right)\right) \\
&\quad + p^2\left(1-\delta^H\right)\left(k-1\right) z_{t+1} + p\left(1-p\right)\left(1-\delta^H\right)\left(k-1\right) z_{t+1} + \left(1-p\right)^2 \delta^H \hat{u}^{HH}_{-z_t} \\
&= p\left(1-\delta^H\right) + p\delta^H\left(\bar{u}_{em} + \underline{u}_{em}\right) + p^2\delta^H\left(\hat{u}_{em} - \left(\bar{u}_{em} + \underline{u}_{em}\right)\right) \\
&\quad + p\left(1-\delta^H\right)\left(k-1\right) z_{t+1} + \left(1-p\right)^2 \delta^H \hat{u}^{HH}_{-z_t}.
\end{aligned}
$$

Recall from (71) that $\bar{u}_{em} + \underline{u}_{em} = p\left(k+1\right)$, so

$$
\begin{aligned}
\hat{u}^{HH}_{-z_t} &= p\left(1-\delta^H\right) + p\delta^H p\left(k+1\right) + p^2\delta^H\left(\hat{u}_{em} - p\left(k+1\right)\right) \\
&\quad + (1-\delta^H)p(k-1)z_{t+1} + \left(1-p\right)^2 \delta^H \hat{u}^{HH}_{-z_t} \\
&= \underbrace{p\left(1-\delta^H\right) + p^2\delta^H\left(\left(k+1\right)\left(1-p\right) + \hat{u}_{em}\right)}_{\equiv c_1} \\
&\quad + \underbrace{(1-\delta^H)p(k-1)}_{\equiv c_2} z_{t+1} + \underbrace{\delta^H\left(1-p\right)^2}_{\equiv d_{hh}} \hat{u}^{H}_{-z_t} \\
&= c_1 + c_2 z_{t+1} + d_{hh}\left(c_1 + c_2 z_{t+2} + d_{hh} u^{HL}_{-z_{t+2}}\right) \\
&= c_1\left(1 + d_{hh} + d_{hh}^2 + \ldots\right) + c_2\left(z_{t+1} + d_{hh}\, z_{t+2} + d_{hh}^2 z_{t+3} + \ldots\right) \\
&= \tfrac{c_1}{1-d_{hh}} + c_2 \sum_{i=0}^{\infty} d_{hh}^i z_{t+1+i} = p + \left(1-\delta^H\right)\alpha + (1-\delta^H) S_{iHH} \qquad (117)
\end{aligned}
$$

$$\text{for } S_{iHH} \equiv p(k-1)\sum_{i=0}^{\infty}\left(\delta^H(1-p)^2\right)^i z_{t+1+i}, \text{ and}$$

$$\alpha \equiv \frac{\frac{c_1}{1-d_{hh}}-p}{\left(1-\delta^H\right)} = \frac{\delta^H(k-1)p^2(1-\delta^H(1-p(2-(3-p)p)))}{(1-\delta^H)\left(1-\delta^H(1-p)^2\right)(1-\delta^H(1-2(1-p)p))}$$

We are now ready to return back to inequality (113) that needs to hold for the result to hold. Take all the terms to the left side and call the resultant function, Q.

$$\begin{aligned} Q := \quad & \tfrac{\mu_t\delta^H}{1-\delta^H}\left(p\left(\hat{u}_{em}-\underline{u}_{em}\right)+(1-p)\left(\bar{u}_{em}-\hat{u}^{HH}_{-z_t}\right)\right)+\tfrac{(1-\mu_t)\delta^H}{1-\delta^H}\left(p-\hat{u}^{HL}_{-z_t}\right) \\ & -\tfrac{\mu_t\delta^L}{1-\delta^L}p\left(\bar{u}^L_{em}-p\right)-\tfrac{\mu_t\delta^L}{1-\delta^L}(1-p)\left(\bar{u}^L_{em}-\hat{u}^{LH}_{-z_t}\right). \end{aligned}$$

Substituting in for payoffs from (69), (71), (117), (116), (114), (115) yields

$$\begin{aligned} Q \;=\; & \mu_t\delta^H p\tfrac{p\left(k\left(1-\delta^H(1-p)^2\right)-\delta^H p^2\right)}{(1-\delta^H(1-2p))(1-\delta^H(1-2(1-p)p))} \\ & +\mu_t\delta^H\tfrac{p(1-p)}{2}\left(\tfrac{2(k-1)}{1-\delta^H(1-p)^2}+\tfrac{k+1}{1-\delta^H+2\delta^H p}-\tfrac{k-1}{1-\delta^H(1-2(1-p)p)}\right) \\ & -\mu_t\delta^H p(1-p)(k-1)\sum_{i=0}^{\infty}\left(\delta^H(1-p)^2\right)^i z_{t+1+i} \\ & +(1-\mu_t)\delta^H p\sum_{i=0}^{\infty}\left(\delta^H(1-p)\right)^i z_{t+1+i}-\tfrac{\mu_t\delta^L pk}{1-\delta^L(1-p)} \\ & +\mu_t\delta^L pk(1-p)\sum_{i=0}^{\infty}\left(\delta^L(1-p)\right)^i z_{t+1+i}. \end{aligned}$$

Observe that for converging series there exists $z_\omega = \frac{\sum d^i_\omega z_i}{\sum d^i_\omega}$ and z_i decreasing, then $d_1 \leq d_2 \Longrightarrow z_1 \geq z_2$ so we can write Q as,

$$\begin{aligned} Q \;=\; & \tfrac{\mu_t\delta^H p^2\left(k\left(1-\delta^H(1-p)^2\right)-\delta^H p^2\right)}{(1-\delta^H(1-2p))(1-\delta^H(1-2(1-p)p))} \\ & +\tfrac{\mu_t\delta^H p(1-p)}{2}\left(\tfrac{2(k-1)}{1-\delta^H(1-p)^2}+\tfrac{k+1}{1-\delta^H+2\delta^H p}-\tfrac{k-1}{1-\delta^H(1-2(1-p)p)}\right) \\ & -\tfrac{\mu_t\delta^L pk}{1-\delta^L(1-p)}+\tfrac{\mu_t\delta^L pk(1-p)}{1-\delta^L(1-p)}z_L-\tfrac{\mu_t\delta^H p(1-p)(k-1)}{1-\delta^H(1-p)^2}\bar{z}_H+\tfrac{(1-\mu_t)\delta^H p}{1-\delta^H(1-p)}\underline{z}_H \end{aligned}$$

where

$$\begin{aligned} z_L \;&=\; \frac{\sum_{i=0}^{\infty}\left(\delta^L(1-p)\right)^i z_{t+1+i}}{\sum_{i=0}^{\infty}\left(\delta^L(1-p)\right)^i} \\ &=\; \left(1-\delta^L(1-p)\right)\sum_{i=0}^{\infty}\left(\delta^L(1-p)\right)^i z_{t+1+i} \in (0, z_t) && (118) \\ \bar{z}_H \;&=\; \left(1-\delta^H(1-p)^2\right)\sum_{i=0}^{\infty}\left(\delta^H(1-p)^2\right)^i z_{t+1+i} \in (0, z_t) && (119) \\ \underline{z}_H \;&=\; \left(1-\delta^H(1-p)\right)\sum_{i=0}^{\infty}\left(\delta^H(1-p)\right)^i z_{t+1+i} \in (0, z_t) && (120) \end{aligned}$$

observe that $\bar{z}_H > \underline{z}_H$ and $0 < z_L, \underline{z}_H, \bar{z}_H \leq 1$.

Next we show that Q is decreasing in δ^L and increasing in δ^H.

$$\frac{\partial Q}{\partial \delta^L} = -\mu_t \tfrac{pk(1-(1-p)z_L)}{(1-\delta^L(1-p))^2} < 0 \text{ since } z_L \in (0,1)\,.$$

Since we want to prove that $Q \geq 0$ for all values of $\delta^L \in (0, \delta^*)$, and Q is decreasing in δ^L, it is enough to prove that $Q \geq 0$ for $\delta^L = \delta^*$. Next we show that Q is increasing in δ^H.

$$\begin{aligned}
\tfrac{\partial Q}{\partial \delta^H} &= \frac{1}{2}p\left(-\mu_t \tfrac{(k-1)(1-2p)}{\left(1-\delta^H+2\delta^H p-2\delta^H p^2\right)^2} + \mu_t \tfrac{(k+1)}{(1-\delta^H(1-2p))^2}\right.\\
&\quad \left.+\ \mu_t \tfrac{2(k-1)(1-p)(1-\bar{z}_H)}{\left(1-\delta^H(1-p)^2\right)^2} + (1-\mu_t)\tfrac{2}{(1-\delta^H(1-p))^2}\underline{z}_H\right)\\
&= \frac{1}{2}p\mu_t(\overbrace{\tfrac{k+1}{(1-\delta^H(1-2p))^2} - \tfrac{(k-1)(1-2p)}{\left(1-\delta^H(1-2p+2p^2)\right)^2}}^{>0 \text{ per claim 7}}\\
&\quad +\mu_t \tfrac{2(k-1)(1-p)(1-\bar{z}_H)}{\left(1-\delta^H(1-p)^2\right)^2} + (1-\mu_t)\tfrac{2}{(1-\delta^H(1-p))^2}\underline{z}_H) > 0. \qquad (121)
\end{aligned}$$

Since Q is increasing in δ^H, it is enough to prove that $Q \geq 0$ for $\delta^H = \underline{\delta}_1^H$. To that end, let $Q^* = Q$ s.t. $\delta^L = \delta^*$ and $\delta^H = \underline{\delta}_1^H = \frac{1}{1-2p+p^2(k+1)}$. Then

$$Q^* = \frac{\mu_t(1-p)(k-1)}{pk}\left(1-\bar{z}_H\right) + \frac{1-\mu_t}{p(k+1)-1}\underline{z}_H + \mu_t(1-p)z_L > 0.$$

Recall that condition (60) implies that $p(k+1) \geq 2$, so the second term of Q^* is positive and bounded, and the $\bar{z}_H, \underline{z}_H, z_L \in (0,1)$, so $Q^* > 0 \Rightarrow Q > 0$ for all $\delta^L \in (0, \delta^*)$ and $\delta^H \in [\underline{\delta}_1^H, 1)$.Furthermore, the constraint is slack, so it follows that we could find equilibria for lower δ^H. Last, let Q^L denote the part of Q that represents ICC^L_{iss} That is,

$$Q^L = \frac{\mu_t \delta^L pk}{1-\delta^L(1-p)} - \frac{\mu_t \delta^L pk\,(1-p)}{1-\delta^L\,(1-p)}z_L.$$

Since $\frac{\partial Q}{\partial \delta^L} < 0 \implies \frac{\partial Q^L}{\partial \delta^L} > 0$, so Q^L is increasing in δ^L as is to be expected. Since $\delta^L \in (0, \delta^*)$, it follows that

$$\begin{aligned}
Q^L &\in \left(Q^L\big|_{\delta^L=0}, Q^L\big|_{\delta^L=\delta^*}\right) = (0, \mu_t(1-(1-p)z_L^*))\\
&= \left(0, \mu_t\left(1-(1-p)\left(1-\delta^*\left(1-p\right)\right)\sum_{i=0}^{\infty}\left(\delta^*\left(1-p\right)\right)^i z_{t+1+i}\right)\right) \text{ by (118)}\\
&= \left(0, \mu_t - \mu_t \tfrac{(1-p)pk}{1+p(k-1)}\sum_{i=0}^{\infty}\left(\tfrac{1-p}{1+p(k-1)}\right)^i z_{t+1+i}\right) \text{ by (5)}\\
&= \left(0, \mu_t - \mu_t pk\sum_{i=0}^{\infty}\tfrac{(1-p)^{i+1}}{(1+(k-1)p)^{i+1}}z_{t+1+i}\right)\\
&= \left(0, \mu_t\left(1-p+p^2\right)\tilde{z}_L\right) \text{ for } \tilde{z}_L \in (0, z_t) \subseteq (0,1]\\
&\subset \left(0, \mu_t\left(1-p+p^2\right)\right).
\end{aligned}$$

That is, the lower bound for z_t is well-defined. Next, let Q^H denote the part of Q that represents

ICC_{iss}^{H}.

$$\begin{aligned} Q^H &= \tfrac{\mu_t \delta^H p^2\left(k\left(1-\delta^H(1-p)^2\right)-\delta^H p^2\right)}{(1-\delta^H(1-2p))(1-\delta^H(1-2(1-p)p))} - \tfrac{\mu_t\delta^H p(1-p)(k-1)}{1-\delta^H(1-p)^2}\bar{z}_H + \tfrac{(1-\mu_t)\delta^H p}{1-\delta^H(1-p)}\underline{z}_H \\ &\quad + \tfrac{\mu_t\delta^H p(1-p)}{2}\left(\tfrac{2(k-1)}{1-\delta^H(1-p)^2} + \tfrac{k+1}{1-\delta^H+2\delta^H p} - \tfrac{k-1}{1-\delta^H(1-2(1-p)p)}\right). \end{aligned}$$

Since $\frac{\partial Q}{\partial \delta^H} > 0 \implies \frac{\partial Q^H}{\partial \delta^H} > 0$, so Q^H is increasing in δ^H as is to be expected. Since $\delta^H \in \left(\underline{\delta}_1^H, 1\right)$, it follows that

$$\begin{aligned} Q^H &\in \left(Q^H\Big|_{\delta^H=\underline{\delta}_1^H}, Q^H\Big|_{\delta^H=1} \right) \\ &= \Bigg(\mu_t \tfrac{k-1+p}{pk} - \mu_t \tfrac{(k-1)(1-p)}{pk}\bar{z}_H + \tfrac{(1-\mu_t)}{p(k+1)-1}\underline{z}_H \,, \\ &\qquad \mu_t\left(\tfrac{k}{2} + \tfrac{(k-1)p^2}{4(1-p)(2-p)}\right) - \mu_t\tfrac{(k-1)(1-p)}{2-p}\bar{z}_H + (1-\mu_t)\underline{z}_H \Bigg). \end{aligned}$$

The simplifications for z's can be found at the end,

$$\begin{aligned} Q^H &\in \Bigg(\mu_t\tfrac{k-1+p}{pk} + \overbrace{\tfrac{(1-\mu_t)}{p(k+1)-1}\underline{z}_H - \mu_t\tfrac{(k-1)(1-p)}{pk}\bar{z}_H}^{\equiv(*)}, \\ &\qquad \underbrace{(1-\mu_t)\underline{z}_H - \mu_t\tfrac{(k-1)(1-p)}{2-p}\bar{z}_H}_{\equiv(**)} + \mu_t\tfrac{4k+3kp^2-6kp-p^2}{4(p-1)(p-2)} \Bigg). \end{aligned} \tag{122}$$

Then there exists $\tilde{z}_{h1}, \tilde{z}_{h2} \in (0, z_t)$ such that

$$\begin{aligned} Q^H &\in \Bigg(\mu_t\tfrac{k-1+p}{pk} + \left(\tfrac{1}{p(k+1)-1} + \mu_t\tfrac{k(1-2p)-(1-p)\left(1+p\left(k^2-1\right)\right)}{pk(p(k+1)-1)}\right)\tilde{z}_{h1}, \\ &\qquad \mu_t\left(\frac{k}{2} + \tfrac{(k-1)p^2}{4(1-p)(2-p)}\right) + \left(1 - \tfrac{\mu_t(k(1-p)+1)}{2-p}\right)\tilde{z}_{h2} \Bigg). \end{aligned}$$

That is, the upper bound for z_t is well-defined. ■

Proof. (Inequality 121):

$$\begin{aligned} \tfrac{k+1}{(1-\delta^H(1-2p))^2} &> \tfrac{(k-1)(1-2p)}{\left(1-\delta^H(1-2p+2p^2)\right)^2} \\ \iff \tfrac{\left(1-\delta^H\left(1-2p+2p^2\right)\right)^2}{(1-\delta^H(1-2p))^2} &> \tfrac{(k-1)(1-2p)}{k+1} \\ \iff \tfrac{1-\delta^H(1-2p)-2\delta^H p^2}{1-\delta^H(1-2p)} &> \sqrt{\tfrac{(k-1)(1-2p)}{k+1}} \\ \iff 1 - \tfrac{2\delta^H p^2}{1-\delta^H(1-2p)} &> \sqrt{\tfrac{(k-1)(1-2p)}{k+1}} \end{aligned} \tag{123}$$

The left-hand side of (123) is minimized at $\delta^H = 1$, so it is enough to verify that

$$\begin{aligned} &1-p > \sqrt{\tfrac{(k-1)(1-2p)}{k+1}} \\ \Longleftarrow\quad &(k+1)\left(1-p\right)^2 - (k-1)(1-2p) > 0 \end{aligned}$$

$$
\begin{aligned}
&\iff \quad p^2k - 4p + p^2 + 2 > 0 \\
&\Longleftarrow \quad p^2k - 4p + p^2 + 2 \geq 2p^2 - 4p + 2 = 2\left(p-1\right)^2 > 0.
\end{aligned}
$$

Therefore, the inequality used in step (121) of the proof of proposition 28 holds. ■

Bibliography

[1] Abdulkadiroğlu, A. and Bagwell K. 2007. Trust, Reciprocity and Favors in Cooperative Relationships. Mimeo.

[2] Abreu, D., Pearce, D. and Stacchetti, E. 1990. Toward a Theory of Discounted Repeated Games with Imperfect Monitoring. Econometrica, 58, 1041–1063.

[3] Durrett, R. 1996. Probability: Theory and Examples.2nd Edition. Duxbury Press.

[4] Eeckhout, J. 2001. Competing Norms of Cooperation. Mimeo.

[5] Fudenberg, D., Levine, D. K. and Maskin, E. 1994. The Folk Theorem in Repeated Games with Imperfect Public Information. Econometrica 62, 997-1039.

[6] Fudenberg, D. and Tirole, J. 1991. Game Theory. The MIT Press.

[7] Ghosh, P. and Ray, D. 1996. Cooperation in Community Interaction without Information Flows. The Review of Economic Studies, Vol. 63, No. 3, pp. 491-519.

[8] Hauser, C. and Hopenhayn, H. 2005. Trading favors: Optimal Exchange and Forgiveness. Mimeo.

[9] Kocherlakota, N.R. 1996. Implications of Efficient Risk Sharing without Commitment. Review of Economic Studies, Vol. 63, No. 4, pp. 595-609.

[10] Mailath, G.J. and Samuelson, L. 1998. Your Reputation Is Who You're Not, Not Who You'd Like to Be. CARESS Working Paper No. 98-11, University of Pennsylvania.

[11] Mailath, G.J. and Samuelson, L. 2001. Who Wants a Good Reputation? Review of Economic Studies, 68(2), 415–441.

[12] Möbius, M. 2001. Trading Favors. Mimeo.

[13] Nowak, M.A. and Sigmund, K. 1998a. The dynamics of indirect reciprocity. J. Theor. Biol. 194, 561-574.

[14] Nowak, M.A. and Sigmund, K. 1998b. Evolution of indirect reciprocity by image scoring. Nature. 393, 573-577.

[15] Okuno-Fujiwara, M. and Postlewaite, A. 1995. Social Norms in Matching Games. Games and Economic Behavior, Vol. 9, pp79-109.

[16] Samuelson, L. 2008. Favors. Mimeo.

[17] Sannikov, Y. 2007. Games with Imperfectly Observable Actions in Continuous Time. Econometrica, Econometric Society, vol. 75(5), pages 1285-1329, 09.

[18] Suzuki, S. and Akiyama, E. 2005. Reputation and the evolution of cooperation in sizable groups. Proc Biol Sci. 272 (1570):1373-7.

[19] Watson, J. 1999. Starting Small and Renegotiation. Journal of Economic Theory 85, 52-90.

[20] Watson, J. 2002. Starting Small and Commitment. Games and Economic Behavior 38: 176-99.

[21] Wolfram Research, Inc. 2009. Numerical Algorithms for Constrained Global Optimization. `http://reference.wolfram.com/mathematica/tutorial/ConstrainedOptimizationGlobalNumerical.html#85183321`, retrieved November 17, 2009.