# Auctioning Emission Allowances

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**Abstract**

We consider Vickrey-Clarke-Groves mechanism and Uniform Price Auction in allocating the emission allowances in the emission trading market. We study the oligopolistic competition in allowance markets and build a supply function equilibrium model a la Klemperer and Meyer (1989). We also study possibility for collusive behaviour and calculate the impacts of the collusion on the efficiency of the allowance allocation as well as auction revenues of the regulator. Wilson (1979) has shown the possibilities for extreme low price equilibria in the uniform price auctions. In our model, the market will consist of two parts, competitive fringe and number of strategic players. The fringe will balance the market and this will guarantee the absence of extreme low price equilibria. We restrict to linear strategies and the demand schedules are fully defined by the single parameter, namely the slope of the demand. With the help of this construction we find unique equilibrium in the numerical model.

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## 1 Introduction

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## 2 Model

We follow the notation by Montero (2008) and consider a model of large number of firms, which are divided in two types of players, namely competitive fringe and number of $n$ strategic firms. The market has aggregate linear inverse demand and demand functions for emission allowances, respectively, as following,

$$P(X) = A - \beta X \quad \text{or} \quad X(P) = \frac{A}{\beta} - \frac{1}{\beta}p \tag{1}$$

where $p$ denotes the price of the emission allowance, $A$ is some positive constant and $\beta$ is the slope parameter of the inverse demand function, and $X$ is the total use of allowances or amount of emissions. First set of players are competitive fringe of firms of total size $1-\theta$ of the market, with aggregate demand,

$$P_f(X_f) = A - \beta_f X_f \quad \text{or} \quad X_f(p) = \frac{A}{\beta_f} - \frac{1}{\beta_f}p \tag{2}$$

where $\beta_f = \frac{1}{1-\theta}\beta$. The relative size of a single fringe firm is so small that we assume it to bid with marginal costs. However, second type of players behaves strategically. We assume $k$ identical strategic firms indexed with $i$ and each having demand for allowances as following,

$$P_i(x_i) = A - \beta_i x_i \quad \text{or} \quad X_i(p) = \frac{A}{\beta_i} - \frac{1}{\beta_i}p \tag{3}$$

where $\beta_i = \frac{k}{\theta}\beta$. Strategic firms may also co-operate when reporting their schedules to the regulator. If this is the case, the co-operative (cartel) firms report one joint demand schedule or one collusive firm reports the aggregate schedule while the others report null schedules. After the auction the collusive firms share the costs and allowances using some allocation mechanism which we assume to be efficient and equal with respect to their cost functions. The joint true demand of the cartel consisting of $m$ strategic firms, leaving $n = k - m$ startegic firms playing individually, can be defined as

$$P_m(x_m) = A - \beta_m x_m \quad \text{or} \quad X_m(p) = \frac{A}{\beta_m} - \frac{1}{\beta_m}p \tag{4}$$

where $\beta_m = \frac{1}{m}\beta_i = \frac{k}{m\theta}\beta$.

The business as usual level of emissions are denoted with $x_i^0$, $x_m^0$ and $X_f^0$ for single startegic firm, cartel and the fringe, respectively. These are summed to $X^0$ for a whole market. The abatement costs the firm faces when reducing emissions from $x_i^0$ to $x_i$ can be characterised by the equation $AC_i(x_i) = \int_{x_i}^{x_i^0} P_i(x)dx$. In our model, we use fixed amount of allowances $L = \delta X^0$ where $0 < (1-\delta) < 1$ is the reduction target from initial emissions.[1] Allowances will be allocated to the markets by different auction designs.

In the auction, each firm (or cartel) reports demand schedule $\hat{X}_i(p)$ to the regulator. The schedules are limited to be linear and strictly decreasing functions and we assume them to have equal constant term $A$, which is common knowledge for all firms. Thus, the reported demand schedules have functional form as $\hat{X}_i(p) = \frac{A}{b_i} - \frac{1}{b_i}p$. The regulator computes total demand and residual supply for every firm by,

$$S_i(p) = L - \hat{X}_{-i}(p) \tag{5}$$

[1] This is one special case in Montero (2008). He studies more general model where supply of allowances is defined by marginal damage function of pollution.

where $\hat{X}_{-i}(p) = \sum_{j \neq i}^{n} \hat{X}_j(p) + X_f(p)$ is the aggregate demand schedules reported by every other firm but firm $i$. The regulator clears the auction by the clearing rule,

$$p = \hat{P}_i(l_i) = S_i^{-1}(l_i) \quad \text{or} \quad l_i = S_i(p) = \hat{X}_i(p) \tag{6}$$

where $l_i$ is the number of emission allowances allocated to firm $i$ and $L = \sum_{i=1}^{n} l_i + L_f$. Assuming $\hat{X}_{-j}(0) \geq L$ (for $j = i, m, f$)[2] we may write the inverse residual supply functions for individual startegic firm and for cartel, given the reports $b_i$ and $b_m$,

$$S_i^{-1}(x_i) = A - \frac{L}{\left(\frac{1}{\beta_f} + \frac{1}{b_m} + \frac{n-1}{b_i}\right)} + \frac{1}{\left(\frac{1}{\beta_f} + \frac{1}{b_m} + \frac{n-1}{b_i}\right)} x_i \tag{7}$$

$$S_m^{-1}(x_m) = A - \frac{L}{\left(\frac{1}{\beta_f} + \frac{n}{b_i}\right)} + \frac{1}{\left(\frac{1}{\beta_f} + \frac{n}{b_i}\right)} x_i \tag{8}$$

## 2.1 VCG mechanism

Montero (2008) introduced a Vickrey-Clarke-Groves mechanism in pollution problem setup. In VCG mechanism, in addition to the auction procedure described above, every firm gets payback after the auction. The amount of payback is defined with the formula

$$\alpha_i = 1 - \frac{\int_0^{l_i} S_i^{-1}(x)dx}{S_i^{-1}(l_i) l_i}. \tag{9}$$

By Proposition 3 in Montero (2008), due to paybacks, it is optimal for each firm to report its true demand curve irrespective of the other firms' reports. Thus the mechanism is implementing the first best in dominant strategies. In equilibrium all firms face the same price and receive optimal amount of allowances, but the final prices $(1 - \alpha_i)p$ differs between firms unless they all have identical demand schedules. The final payment in the auction for each firm is equal to the externality firm is causing to other firms. If we had a emission damage function in the model, like in Montero (2008), the externality would consist of two factors. First, the total damage of its own pollution and secondly the external cost due to the increase in allowance price. As the number of strategic firms increases, the $\alpha_i$ will be close to zero and the final allowance price is equal to the Pigouvian tax.

Due to the truthfull reporting, VCG mechanism implements efficient allocation of allowances and thus cost minimising solution for pollution control. However, the

[2] This will hold at least when $X_f^0 \geq L \Rightarrow 1 - \theta \geq \delta$. If, on the other hand $\hat{X}_{-j}(0) < L$ (for some $j = i, m, f$), then $S_j^{-1}(x_j) = 0$ for $0 \leq x_j < L - \hat{X}_{-j}(0)$.

revenue from the auction will be smaller, the more influence the indidual firms have to the equilibrium price.

The first best allocation holds even if there is collusion in the auction. Still, it is dominant strategy to report truthfully and the outcome will be the first best. However, compared to the competitive market, the payback to the cartel is greater due to the greater impact on aggregate demand and larger residual supply in every price. Under collusion the auction revenue of the regulator will be smaller than in the more competitive market structure.

In our model the payback for fringe is zero, and the paybacks for strategic players are $\alpha_i = \frac{1}{2}\frac{\theta\delta}{(n+m-\theta)(1-\delta)}$ and for cartel $\alpha_m = \frac{1}{2}\frac{\theta\delta}{\left(\frac{n+m}{m}-\theta\right)(1-\delta)}$ (see the derivation in Appendix). Keeping the number of strategic palyers, $k = n + m$, fixed, the payback for cartel is increasing in the size of the collusive firms $m$ , while the relative costs of emission reduction or equilibrium price of emission allowances will not change. Thus, it is profitable for cartel members to increase the size of the cartel as high as possible.

## 2.2 Uniform price auction

In the setting of uniform price auction it is not more profitable to report demands truhfully as long as firms may influence the auction price. We adopt symmetric strategies for all individual strategic players $i = 1, ..., n$. The objective of the firm is to minimze its cost with respect to the reported $b_i$,

$$\min_{b_i} TC_i = \int_{l_i(b_i)}^{x_i^0} P_i(x)dx + l_i(b_i)S_i^{-1}(l_i(b_i)) \tag{10}$$

First order condition for single strategic firm is given by,

$$-P_i(l_i)\frac{dl_i}{db_i} + S_i^{-1}(l_i)\frac{dl_i}{db_i} + l_i\frac{dS_i^{-1}(l_i)}{dl_i}\frac{dl_i}{db_i} = 0 \tag{11}$$

$$P_i(l_i) - l_i\frac{dS_i^{-1}(l_i)}{dl_i} = S_i^{-1}(l_i)$$

$$\Rightarrow \hat{P}_i(l_i) = P_i(l_i) - l_i\frac{dS_i^{-1}(l_i)}{dl_i}.$$

Adopting symmetric strategies we can solve the optimal symmetric best responce strategy for all individual strategic firms against the cartel firm equalising the reported schedule with the optimal strategy from F.O.C ,

$$\begin{aligned} A - b_i l_i &= A - \beta_i l_i - \frac{1}{\left(\frac{1}{\beta_f} + \frac{1}{b_m} + \frac{n-1}{b_i}\right)} l_i \\ b_i &= \beta_i + \frac{1}{\left(\frac{1}{\beta_f} + \frac{1}{b_m} + \frac{n-1}{b_i}\right)} \\ &= \beta_i + \frac{\beta_f}{1 + \frac{\beta_f}{b_m} + \frac{(n-1)\beta_f}{b_i}} \end{aligned}$$

Solving this for $b_i$ gives a second order equation,

$$\underbrace{\left(1 + \frac{\beta_f}{b_m}\right)}_{B_1} b_i^2 + \underbrace{\left[(n-2)\beta_f - \left(1 + \frac{\beta_f}{b_m}\right)\beta_i\right]}_{B_2} b_i - \underbrace{(n-1)\beta_f \beta_i}_{B_3} = 0.$$

Because $b_i \geq \beta_i$ we get the best responce strategy for player $i$ as,

$$\begin{aligned} BR_i(b_m) &= \frac{-B_2 + \sqrt{B_2^2 + 4B_1 B_3}}{2B_1} \\ &= \frac{\left(1 + \frac{\beta_f}{b_m}\right)\beta_i - (n-2)\beta_f}{2\left(1 + \frac{\beta_f}{b_m}\right)} \\ &\quad + \frac{\sqrt{\left[(n-2)\beta_f - \left(1 + \frac{\beta_f}{b_m}\right)\beta_i\right]^2 + 4(n-1)\left(1 + \frac{\beta_f}{b_m}\right)\beta_f \beta_i}}{2\left(1 + \frac{\beta_f}{b_m}\right)}. \end{aligned} \tag{12}$$

This is increasing function in $b_m$, which can be seen in numeric examples.[3]

Similarly, we can write the best responce strategy for the cartel, knowing all the individual players are using symmetric strategies, as,

$$\begin{aligned} A - b_m l_m &= A - \beta_m l_m - \frac{1}{\left(\frac{1}{\beta_f} + \frac{n}{b_i}\right)} l_m \\ BR_m(b_i) &= \beta_m + \frac{1}{\left(\frac{1}{\beta_f} + \frac{n}{b_i}\right)} \\ &= \beta_m + \frac{\beta_f}{1 + \frac{n\beta_f}{b_i}}. \end{aligned} \tag{13}$$

[3]The function $BR_i'(b_m) > 0$, if $\beta_i > \frac{(n-2)}{2(n-1)}\left(\frac{b_m}{(b_m+\beta_f)}\right)\left(\sqrt{B_2^2 + 4B_1 B_3} - B_2\right)$. We derive this condition in Appendix.

which is strictly increasing in $b_i$ and the difference between reported and true value of the slope parameter $(b_m - \beta_m)$ lies between $0 \leq \frac{\beta_f}{1+\frac{n\theta}{(1-\theta)(n+m)}} \leq (b_m - \beta_m) \leq \beta_f$.

The $BR_i$ function is derived similarly as in Klemperer and Meyer (1989) where they define unique supply function equilibrium using the exogenous uncertainty in industry demand (which is analogous to the residual supply for strategic players $S_k(p) = L - X_f(p)$ in our model). Without uncertainty, they show that there exists infinite number of supply functions which satisfy the sufficient and necessary conditions for the optimum. The candidate supply function for firm has to go through the point along firm's residual demand and in that point firm's marginal revenue equals its marginal costs. The candidate function must also have the slope at the optimal point such that the point is profit maximizing along the other firms' residual demands. In other words, there are infinite number of functions which satisfy these conditions. In our model, we restrict the supply functions to be linear and to have fixed constant parameter $A$. This is common knowledge to all players. Thus, our supply functions are fully defined for the whole support with single parameter, the slope of the reported supply. By this construction, we define the unique supply function equilibrium if we find unique solution for the slopes. The solution can be found at the point where the two best responce functions cross.

Figure 1 illustrates the difference between the VCG mechanism and the uniform price auction. In the figure, the market has been split in two equal sized parts, namely in the parts of fringe and single strategic cartel. The cartel and the fringe have both identical aggregate demand functions $P_f(x_f) = P_m(x_m)$ and hence the initial emissions are also equal $x_f^0 = x_m^0$.

In the VCG mechanism (Figure 1A), all the firms give true reports and the allowances are allocated in cost efficient way. The abatement costs for fringe and for cartel are the triangle $AC_f = AC_m \equiv \triangle(l_m a x_m^0)$. In the auction they both pay first the amount of $R_f = p^* l_f$, but the cartel gets paybacks amount equal to the triangle $\alpha_m p^* l_m \equiv \triangle(0 p^* l_m)$.

In the uniform price auction (Figure 1B), the cartel reduces its demand and reports schedule $\hat{P}_m(x_m)$ which lies below the true demand function in every positive $x_m$. Due to the demand reduction, cartel receives allowances amount which is strictly less than in the VCG mechanism and the abatement costs, $AC_m \equiv \triangle(l_m b x_m^0)$, are much higher than in the cost minimising solution. However, the equilibrium price of the allowances is much lower than in the first best price and the auction revenue from cartel is only $R_m = pl_m$. Interestingly, strategic behavior of the cartel makes the fringe firms strictly better off because the allowance price is lowered. The abatement costs of the fringe are reduced to $AC_f \equiv \triangle(l_f d x_m^0)$ and the regulator collects revenue from the fringe amount of $R_f = pl_f$. The total costs of the fringe are lower than in the VCG mechanism and they are lower than the total costs of the cartel. Comparing the results of the example drawn in the Figure 1, we may conclude that the VCG mechanism is strictly better from the regulator's point of view, if the objective of the regulator is to achieve efficient allocation of allowances and to maximize the revenue in the auction. Total abatement costs are minimised and the revenues are larger than in the uniform price auction[4].

(*Figure 1 in here*)

*Figure 1. VCG Mechanism (A) and Uniform Price Auction (B) with single cartel and fringe of equal size.*

# 3 Simulation

To describe the model we run some numerical examples. In the examples we use fixed values for following parameters: $A = 100$, $\beta = 1$, $\delta = 0.5$ and $\theta = 0.5$. In the first exercise, we consider a game where there are multiple cartels of equal size playing against each other. In other words, we keep $m = 0$ fixed and let $n$ vary between 1 and 10. The second, single cartel game is exactly the game described in the Chapter 2. Now we keep the total number of strategic players fixed ($k = 10$) but let the number of cartel members $m$ vary between 2 and 10.

## 3.1 Multiple cartel game

In the multiple cartel game, we look what happens when the size of strategic players increases keeping the relative size of oligopolistic market fixed. The motivation behind this is to check if it is profitable for firms to join with each other or co-operate under different auction mechanism and if this is the case, what are the consequences to the efficiency and auction revenues. We consider multiple players in symmetric oligopolistic game where the size of players may differ. Size of the players is getting larger if the firms are co-operating and form cartels in the auction. There might also be large companies owning several different installations and the cartel, as it is described in our model, can be seen just giving coordinated demand schedules inside the company. In the model, the oligopolistic part of the market is divided in one to ten shares and results are calculated for all of these possibilities. For example, we may consider ten strategic firms. They may start to negotiate with one of the other firms to coordinate their bids and constitute a collusion. If all of the ten original firms find a collusive partner, the market would consist of five equal sized players. The results, normalised to the case of ten strategic firms, are reported in Table 1 (in Appendix) and, in the case of uniform price auction, in Figure 2 (red graph with triangle nodes). Results show that the larger are the cartels the less are the total costs in both auction designs. If it is possible to increase the size of the cartel as high as possible, we end up with the market structure of one big cartel and fringe of competitive firms. This case was already described above with Figure 1.

[4]This can be seen in the first column in Table 1 or in the last column in Table 2 (in Appendix).

(*Figure 2 in here)*

*Figure 2. Total costs of individual strategic firm and cartel member in the single and multiple cartel games in the uniform price auction.*

## 3.2 Single cartel game

Now, the results from multiple cartel game indicated that the size of the cartel or coordination of the bidding schedules would increase as high as possible. This result will hold in VCG mechanism, but in the uniform price setting the story is not reached its end. Next we consider the case where a single member of the cartel could deviate from the collusive agreement and start playing as individual oligopolistic player. We study the case of ten strategic firms and look for incentives for a single member of the cartel to deviate from the agreement as the size of the cartel varies. In Figure 3 are represented the best responce functions derived in Chapter 2 for individual strategic firm and for cartel in cases of $m = 2$, 5 and 9. The best responce functions cross each other once and only once, so we have unique solution in our game with given parameter values and model construction.

(*Figure 3 in here)*

*Figure 3. Best response functions for different number of cartel firms:* $m = 2$ *(Figure A),* $m = 5$ *(Figure B) and* $m = 9$ *(Figure C).*

In Figure 2, the total costs of the cartel member are described with the black and spotted curve of the different size of a cartel. With the green curve (with square nodes) are described the total costs of the deviating firm after the deviation from the cartel agreement. Starting from one big cartel of ten members, we see that it is profitable to deviate from the cartel and leave other nine firms in the cartel. Total costs are lower when single firm is playing against the cartel of nine firms than the costs were if the firm would stay in ten firms collusion. And again in the case of nine cartel members, it is again profitable to leave the agreement. If the cartel is weak in a sense that deviation is easy, there will be firms leaving the agreement until the size of the cartel is three firms.[5]

As a result, we can conclude that if the cartel agreements are weak and there are not too few strategic firms, the uniform price auction may be approximately equal to the competitive market. Thus, the equilibrium and the allocation of allowances are almost first best and the auction revenues are almost as great as in the competitive market.

If, on the other hand, firms can agree on binding collusive behaviour, the uniform price auction will not offer an efficient allocation and the auction revenues will be relatively low.

[5]This result is irrelevant of the number of strategic firms. The optimal cartel size is three no matter how many strategic firms there are.

Using VCG mechanism the regulator can always guarantee the efficient allocation but the auction revenues will be decreasing in increasing collusive behaviour.

# 4 Conclusions

TBA.

# 5 Appendix

*Table 1. Results of the multiple cartel game.*

*(Table 1. in here)*

*Table 2. Results of the single cartel game.*

*(Table 2. in here)*

**Derivation of payback functions.** The residual supply function for strategic firm can be simplified as,

$$
\begin{aligned}
S_i^{-1}(l_i) &= A - \frac{L}{\left(\frac{1}{\beta_f} + \frac{1}{\beta_m} + \frac{n-1}{\beta_i}\right)} + \frac{1}{\left(\frac{1}{\beta_f} + \frac{1}{\beta_m} + \frac{n-1}{\beta_i}\right)} l_i \\
&= A - \frac{L}{\left(\frac{1-\theta}{\beta} + \frac{m\theta}{k\beta} + \frac{(n-1)\theta}{k\beta}\right)} + \frac{1}{\left(\frac{1-\theta}{\beta} + \frac{m\theta}{k\beta} + \frac{(n-1)\theta}{k\beta}\right)} l_i \\
&= A - \frac{\beta L}{\left(1-\theta + \frac{m+n-1}{n+m}\theta\right)} + \frac{1}{\left(1-\theta + \frac{m+n-1}{n+m}\theta\right)} l_i \\
&= A - \frac{\beta L}{\left(\frac{(1-\theta)(n+m)+(n+m)\theta-\theta}{n+m}\right)} + \frac{\beta}{\left(\frac{(1-\theta)(n+m)+(n+m)\theta-\theta}{n+m}\right)} l_i \\
&= A - \frac{\beta L}{\left(\frac{n+m-\theta}{n+m}\right)} + \frac{\beta}{\left(\frac{n+m-\theta}{n+m}\right)} l_i \\
&= A - \frac{n+m}{n+m-\theta}\beta L + \frac{n+m}{n+m-\theta}\beta l_i \\
&= A - \frac{n+m}{n+m-\theta}\delta A + \frac{n+m}{n+m-\theta}\beta l_i \\
&= \left(\frac{(n+m)(1-\delta)-\theta}{n+m-\theta}\right) A + \frac{(n+m)\beta}{n+m-\theta} l_i.
\end{aligned}
$$

Equalising this with the true demand function gives the first best solution,

$$\begin{aligned}
P_i(l_i) &= S_i^{-1}(l_i) \\
A - \frac{n+m}{\theta}\beta l_i &= A - \frac{n+m}{n+m-\theta}\beta L + \frac{n+m}{n+m-\theta}\beta l_i \\
\frac{1}{\theta} l_i &= \frac{1}{n+m-\theta} L - \frac{1}{n+m-\theta} l_i \\
l_i + \frac{n+m-\theta}{\theta} l_i &= L \\
l_i &= \frac{\theta}{(n+m)} L
\end{aligned}$$

We may also make the same for cartel,

$$\begin{aligned}
S_m^{-1}(l_m) &= A - \frac{L}{\left(\frac{1}{\beta_f} + \frac{n}{\beta_i}\right)} + \frac{1}{\left(\frac{1}{\beta_f} + \frac{n}{\beta_i}\right)} l_m \\
&= A - \frac{L}{\left(\frac{1-\theta}{\beta} + \frac{n\theta}{k\beta}\right)} + \frac{1}{\left(\frac{1-\theta}{\beta} + \frac{n\theta}{k\beta}\right)} l_m \\
&= A - \frac{\beta L}{\left(1-\theta + \frac{n\theta}{n+m}\right)} + \frac{\beta}{\left(1-\theta + \frac{n\theta}{n+m}\right)} l_m \\
&= A - \frac{\beta L}{\left(\frac{(1-\theta)(n+m)+n\theta}{n+m}\right)} + \frac{\beta}{\left(\frac{(1-\theta)(n+m)+n\theta}{n+m}\right)} l_m \\
&= A - \frac{n+m}{n+m-m\theta}\beta L + \frac{n+m}{n+m-m\theta}\beta l_m \\
&= \left(\frac{(n+m)(1-\delta) - m\theta}{n+m-m\theta}\right) A + \frac{(n+m)\beta}{n+m-m\theta} l_m \\
&= A - \frac{1}{1-\frac{m\theta}{n+m}}\beta L + \frac{1}{1-\frac{m\theta}{n+m}}\beta l_m
\end{aligned}$$

$$\begin{aligned}
P_m(l_m) &= S_m^{-1}(l_m) \\
A - \frac{n+m}{m\theta}\beta l_m &= A - \frac{n+m}{n+m-m\theta}\beta L + \frac{n+m}{n+m-m\theta}\beta l_m \\
l_m + \frac{n+m-m\theta}{m\theta} l_m &= L \\
l_m &= \frac{m\theta}{n+m} L.
\end{aligned}$$

Finaly, the paybacks in VCG-machanism are derived using

$$1 - \frac{\int\limits_0^l (a-bx)\,dx}{(a-bl)\,l} = 1 - \frac{al - \frac{1}{2}bl^2}{al - bl^2} = 1 - \frac{al - bl^2 + \frac{1}{2}bl^2}{al - bl^2} = \frac{1}{2}\frac{bl}{a-bl}.$$

Now the payback for individual firm is,

$$
\begin{aligned}
\alpha_i(l_i) &= \frac{1}{2}\frac{\frac{n+m}{n+m-\theta}l_i}{\frac{A}{\beta}-\frac{n+m}{n+m-\theta}L+\frac{n+m}{n+m-\theta}l_i} \\
&= \frac{1}{2}\frac{\frac{n+m}{n+m-\theta}\frac{\theta}{n+m}L}{\frac{A}{\beta}-\frac{n+m}{n+m-\theta}L+\frac{n+m}{n+m-\theta}\frac{\theta}{n+m}L} \\
&= \frac{1}{2}\frac{\frac{\theta}{n+m-\theta}L}{\frac{A}{\beta}-L} \\
&= \frac{1}{2}\frac{\frac{\theta\delta}{n+m-\theta}}{1-\delta} \\
&= \frac{1}{2}\frac{\theta\delta}{(n+m-\theta)(1-\delta)}
\end{aligned}
$$

and for cartel,

$$
\begin{aligned}
\alpha_m &= \frac{1}{2}\frac{\frac{n+m}{n+m-m\theta}l_m}{\frac{A}{\beta}-\frac{n+m}{n+m-m\theta}L+\frac{n+m}{n+m-m\theta}l_m} \\
&= \frac{1}{2}\frac{\frac{n+m}{n+m-m\theta}\frac{m\theta}{n+m}L}{\frac{A}{\beta}-\frac{n+m}{n+m-m\theta}L+\frac{n+m}{n+m-m\theta}\frac{m\theta}{n+m}L} \\
&= \frac{1}{2}\frac{\frac{m\theta}{n+m-m\theta}L}{\frac{A}{\beta}-\frac{n+m-m\theta}{n+m-m\theta}L} \\
&= \frac{1}{2}\frac{m\theta\delta}{(n+m-m\theta)(1-\delta)}.
\end{aligned}
$$

**The sign of the best responce function of the individual strategic firm.** In the uniform price auction the solution for the best responce function of strategic firm can be found solving the second order equation.

$$
\underbrace{\left(1+\frac{\beta_f}{b_m}\right)}_{B_1}b_i^2+\underbrace{\left[(n-2)\beta_f-\left(1+\frac{\beta_f}{b_m}\right)\beta_i\right]}_{B_2}b_i-\underbrace{(n-1)\beta_f\beta_i}_{B_3}=0
$$

Thus, the derivative of the function $BR_i(b_m)$ is

$$
\begin{aligned}
BR_i'(b_m) &= \frac{\left[-B_2' + \frac{1}{2}\left(B_2^2 + 4B_1B_3\right)^{-\frac{1}{2}}\left[2B_2B_2' + 4B_1'B_3\right]\right] B_1}{2B_1^2} \\
&\quad - \frac{\left[-B_2 + \sqrt{B_2^2 + 4B_1B_3}\right] B_1'}{2B_1^2} \\
&= \frac{1}{2B_1^2}\left[\left(B_2B_1' - B_2'B_1\right) + \frac{B_2B_2'B_1 + 2B_1'B_3B_1 - \left(B_2^2 + 4B_1B_3\right)B_1'}{\sqrt{B_2^2 + 4B_1B_3}}\right] \\
&= \frac{1}{2B_1^2}\left[\left(B_2B_1' - B_2'B_1\right) + \frac{B_2B_2'B_1 + 2B_1'B_3B_1 - B_2^2B_1' - 4B_1B_3B_1'}{\sqrt{B_2^2 + 4B_1B_3}}\right] \\
&= \frac{1}{2B_1^2}\left[\left(B_2B_1' - B_2'B_1\right) + \frac{B_2\left(B_2'B_1 - B_2B_1'\right) - 2B_1B_3B_1'}{\sqrt{B_2^2 + 4B_1B_3}}\right] \\
&= \frac{1}{2B_1^2}\left[\left(B_2B_1' - B_2'B_1\right)\left[1 - \frac{B_2}{\sqrt{B_2^2 + 4B_1B_3}}\right] - \frac{2B_1B_3B_1'}{\sqrt{B_2^2 + 4B_1B_3}}\right] \\
&= \frac{1}{2B_1^2}\frac{\left[\left(B_2 + \beta_iB_1\right)\left(\sqrt{B_2^2 + 4B_1B_3} - B_2\right) - 2B_1B_3\right]B_1'}{\sqrt{B_2^2 + 4B_1B_3}}
\end{aligned}
$$

Because $B_2' = -\beta_iB_1' > 0$, the derivative $BR_i'(b_m) > 0$ if $\left(B_2 + \beta_iB_1\right)\left(\sqrt{B_2^2 + 4B_1B_3} - B_2\right) - 2B_1B_3 < 0$,

$$
\begin{aligned}
2B_3 &> \left(\frac{B_2}{B_1} + \beta_i\right)\left(\sqrt{B_2^2 + 4B_1B_3} - B_2\right) \\
\beta_i &> \frac{(n-2)}{2(n-1)}\left(\frac{b_m}{\left(b_m + \beta_f\right)}\right)\left(\sqrt{B_2^2 + 4B_1B_3} - B_2\right).
\end{aligned}
$$

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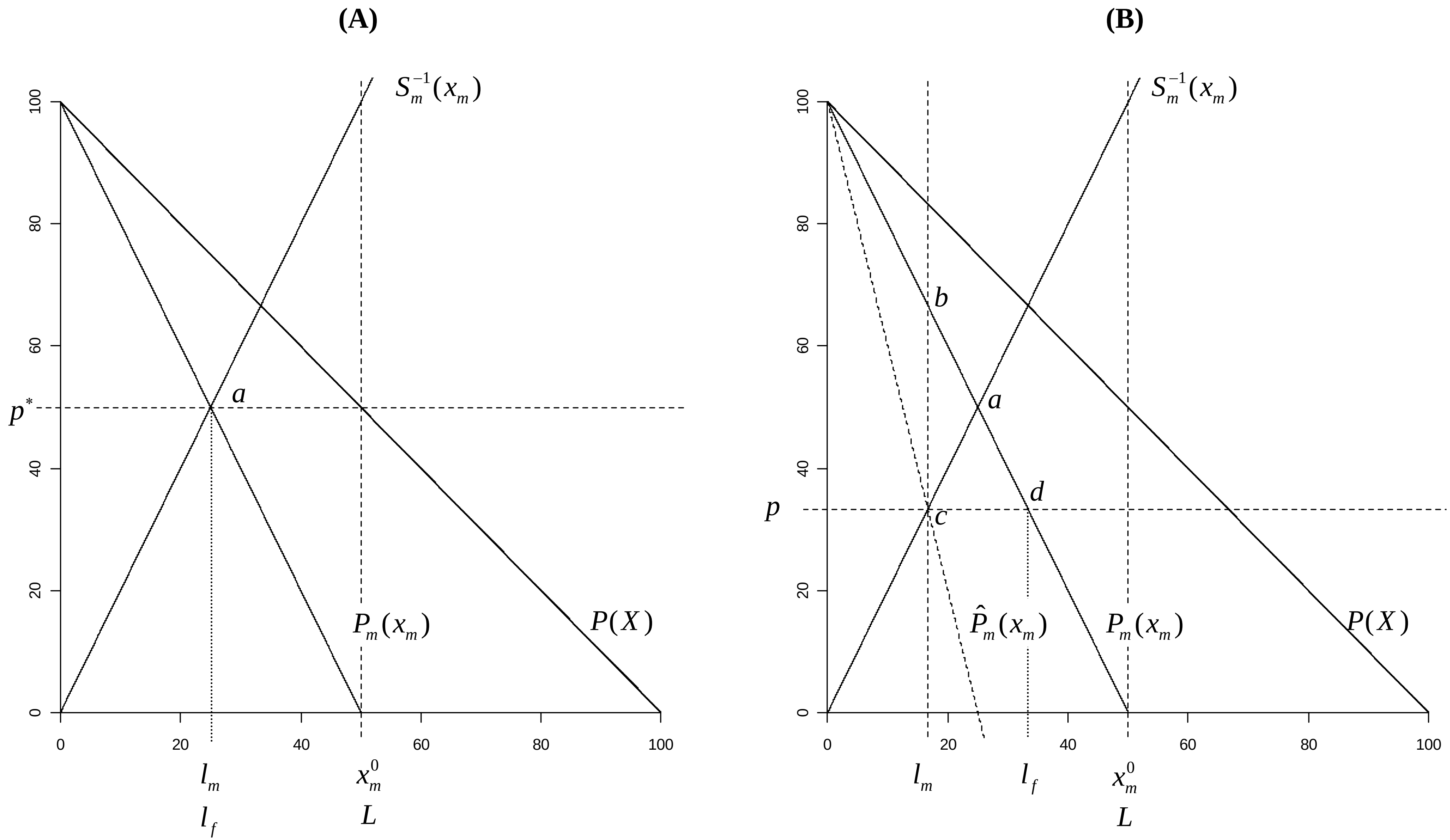


*Figure 1. VCG Mechanism (A) and Uniform Price Auction (B) with single cartel and fringe of equal size.*

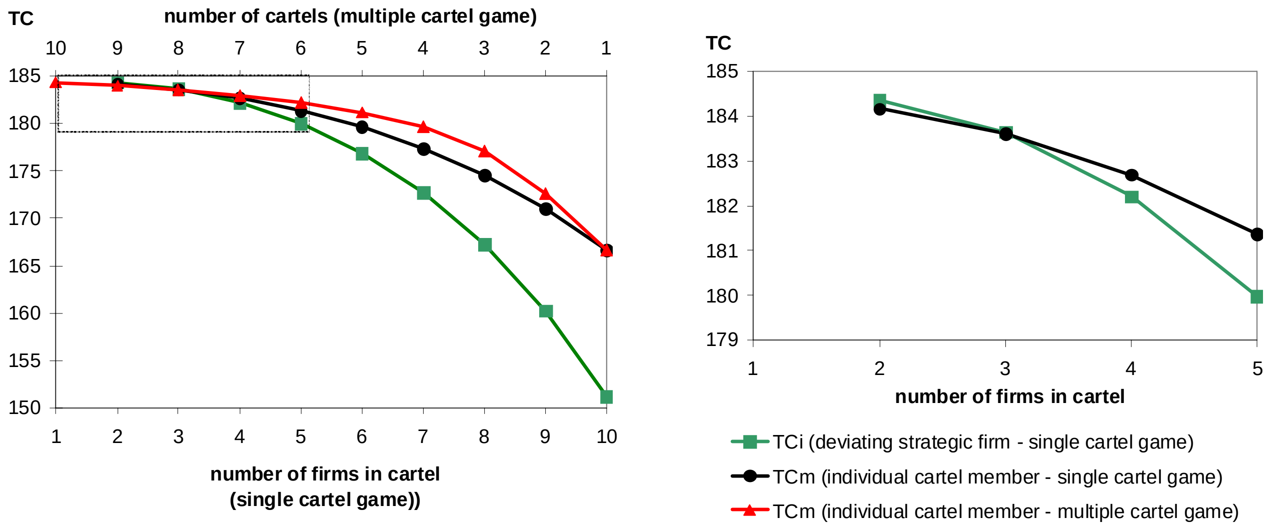


*Figure 2. Total costs of individual strategic firm and cartel member in one and multiple cartel games in the uniform price auction.*

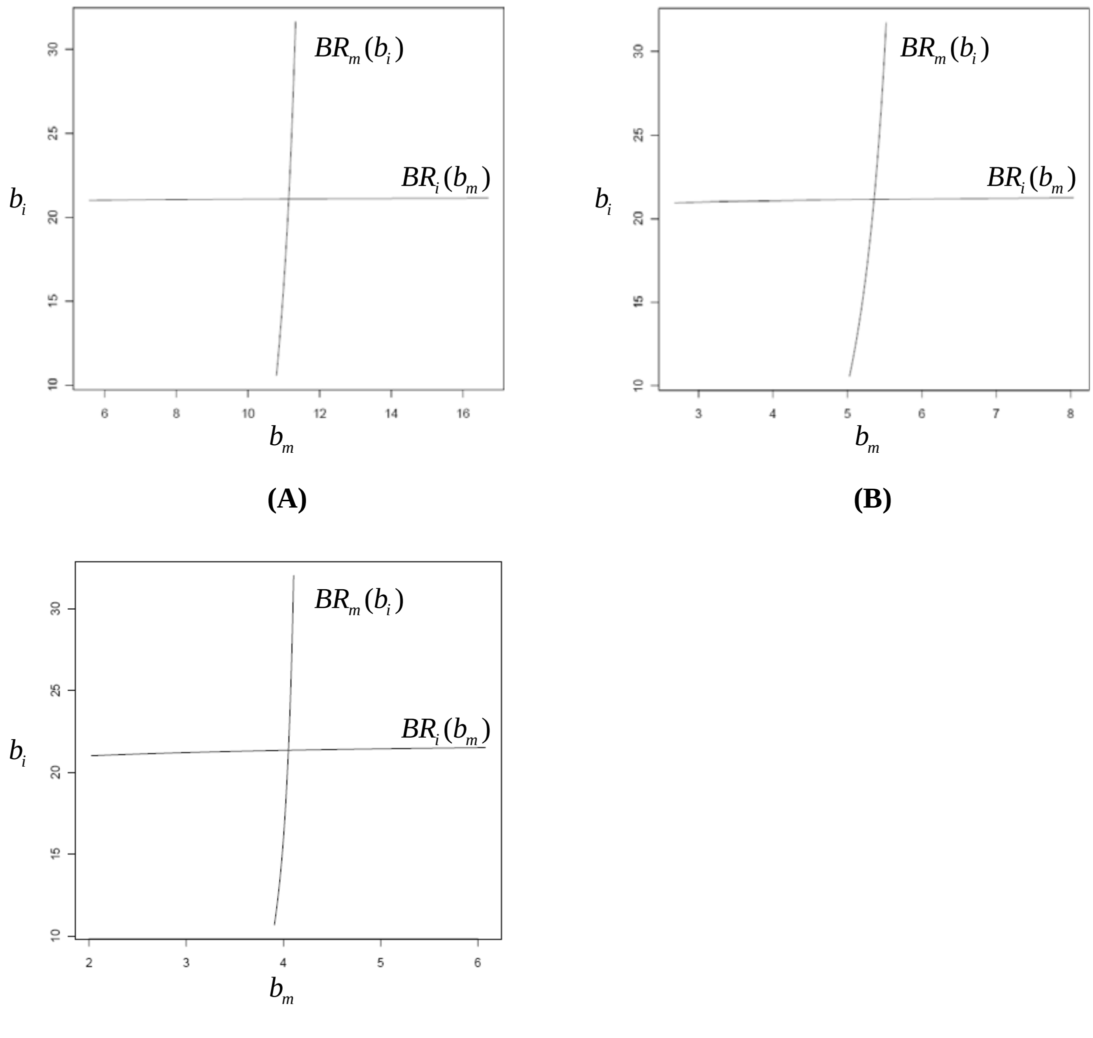


*Figure 3. Best response functions for different number of cartel firms: $m = 2$ (Figure A), $m = 5$ (Figure B) and $m = 9$ (Figure C).*

*Table 1. Results of the multiple cartel game.*

| **Number of cartels** | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Initial Emissions ($x^0$) | | | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| **Uniform Price Auction** | Price | p | **33.33** | 42.26 | 45.14 | 46.48 | 47.25 | 47.74 | 48.09 | 48.34 | 48.53 | 48.69 |
| | Fringe | I | **3.33** | 2.89 | 2.74 | 2.68 | 2.64 | 2.61 | 2.60 | 2.58 | 2.57 | 2.57 |
| | | AC | **27.78** | 44.66 | 50.94 | 54.01 | 55.81 | 56.98 | 57.81 | 58.42 | 58.89 | 59.26 |
| | | TC | **138.89** | 166.67 | 174.76 | 178.39 | 180.43 | 181.73 | 182.62 | 183.28 | 183.78 | 184.17 |
| | Cartel member | I | **1.67** | 2.11 | 2.26 | 2.32 | 2.36 | 2.39 | 2.40 | 2.42 | 2.43 | 2.43 |
| | | AC | **111.11** | 83.33 | 75.24 | 71.61 | 69.57 | 68.27 | 67.38 | 66.72 | 66.22 | 65.83 |
| | | TC | **166.67** | 172.65 | 177.12 | 179.63 | 181.19 | 182.24 | 182.99 | 183.55 | 183.99 | 184.35 |
| | Total | AC | **1388.89** | 1279.92 | 1261.80 | 1256.19 | 1253.79 | 1252.55 | 1251.83 | 1251.38 | 1251.08 | 1250.86 |
| | | R | **1666.67** | 2113.25 | 2257.08 | 2324.08 | 2362.37 | 2387.06 | 2404.27 | 2416.94 | 2426.66 | 2434.35 |
| | | TC | **3055.56** | 3393.16 | 3518.88 | 3580.27 | 3616.16 | 3639.61 | 3656.10 | 3668.32 | 3677.74 | 3685.21 |
| **VCG Mechanism** | Price | p* | **50** | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 50 |
| | Fringe | I | **2.50** | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 |
| | | AC | **62.50** | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 |
| | | TC | **187.50** | 187.50 | 187.50 | 187.50 | 187.50 | 187.50 | 187.50 | 187.50 | 187.50 | 187.50 |
| | Cartel member | I | **2.50** | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 |
| | | AC | **62.50** | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 |
| | | TC | **125.00** | 166.67 | 175.00 | 178.57 | 180.56 | 181.82 | 182.69 | 183.33 | 183.82 | 184.21 |
| | Total | AC | **1250.00** | 1250.00 | 1250.00 | 1250.00 | 1250.00 | 1250.00 | 1250.00 | 1250.00 | 1250.00 | 1250.00 |
| | | R | **1875.00** | 2291.67 | 2375.00 | 2410.71 | 2430.56 | 2443.18 | 2451.92 | 2458.33 | 2463.24 | 2467.11 |
| | | TC | **3125.00** | 3541.67 | 3625.00 | 3660.71 | 3680.56 | 3693.18 | 3701.92 | 3708.33 | 3713.24 | 3717.11 |

*Table 2. Results of the single cartel game.*

| **Number of cartel members** | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Initial Emissions ($x^0$) | | | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| **Uniform Price Auction** | Price | p | 48.69 | 48.41 | **47.86** | 47.01 | 45.84 | 44.31 | 42.38 | 39.98 | 37.01 | 33.33 |
| | Fringe | I | 2.57 | 2.58 | **2.61** | 2.65 | 2.71 | 2.78 | 2.88 | 3.00 | 3.15 | 3.33 |
| | | AC | 59.26 | 58.59 | **57.25** | 55.24 | 52.53 | 49.09 | 44.91 | 39.96 | 34.24 | 27.78 |
| | | TC | 184.17 | 183.47 | **182.02** | 179.79 | 176.66 | 172.48 | 167.01 | 159.94 | 150.79 | 138.89 |
| | Strategic firm | I | 2.43 | 2.45 | **2.47** | 2.51 | 2.56 | 2.63 | 2.72 | 2.82 | 2.95 | 0.00 |
| | | AC | 65.83 | 65.19 | **63.93** | 62.02 | 59.44 | 56.18 | 52.20 | 47.46 | 41.94 | 0.00 |
| | | TC | 184.35 | 183.64 | **182.21** | 179.99 | 176.87 | 172.71 | 167.28 | 160.26 | 151.18 | 0.00 |
| | Cartel member | I | 0.00 | 2.32 | **2.21** | 2.11 | 2.02 | 1.94 | 1.86 | 1.79 | 1.73 | 1.67 |
| | | AC | 0.00 | 72.04 | **77.90** | 83.45 | 88.69 | 93.66 | 98.36 | 102.82 | 107.07 | 111.11 |
| | | TC | 0.00 | 184.16 | **183.61** | 182.69 | 181.37 | 179.61 | 177.35 | 174.52 | 171.01 | 166.67 |
| | Total | AC | 1250.86 | 1251.53 | **1253.75** | 1258.29 | 1265.96 | 1277.60 | 1294.23 | 1317.11 | 1347.91 | 1388.89 |
| | | R | 2434.35 | 2420.58 | **2392.80** | 2350.30 | 2291.86 | 2215.68 | 2119.22 | 1999.00 | 1850.30 | 1666.67 |
| | | TC | 3685.21 | 3672.11 | **3646.54** | 3608.59 | 3557.82 | 3493.28 | 3413.45 | 3316.12 | 3198.21 | 3055.56 |
| **VCG Mechanism** | Price | p* | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 50 | **50** |
| | Fringe | I | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | **2.50** |
| | | AC | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | **62.50** |
| | | TC | 187.50 | 187.50 | 187.50 | 187.50 | 187.50 | 187.50 | 187.50 | 187.50 | 187.50 | **187.50** |
| | Strategic firm | I | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | **0.00** |
| | | AC | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | **0.00** |
| | | TC | 184.21 | 184.21 | 184.21 | 184.21 | 184.21 | 184.21 | 184.21 | 184.21 | 184.21 | **0.00** |
| | Cartel member | I | 0.00 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | 2.50 | **2.50** |
| | | AC | 0.00 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | 62.50 | **62.50** |
| | | TC | 0.00 | 180.56 | 176.47 | 171.88 | 166.67 | 160.71 | 153.85 | 145.83 | 136.36 | **125.00** |
| | Total | AC | 1250.00 | 1250.00 | 1250.00 | 1250.00 | 1250.00 | 1250.00 | 1250.00 | 1250.00 | 1250.00 | **1250.00** |
| | | R | 2467.11 | 2459.80 | 2443.89 | 2417.76 | 2379.39 | 2326.13 | 2254.55 | 2160.09 | 2036.48 | **1875.00** |
| | | TC | 3717.11 | 3709.80 | 3693.89 | 3667.76 | 3629.39 | 3576.13 | 3504.55 | 3410.09 | 3286.48 | **3125.00** |