## Wolinsky's Price Signals Quality

This section is based on Wolinsky (1983). There is a continuum of consumers with identical preferences.

$$u = \theta v - p$$

and the monopolist can provide low and high quality: $v = \{0, 1\}$ at cost $0 < c_0 < c_1$.

The monopolist selects price and quality simultaneously. Assume that $\theta > c_1$ so that it is socially efficient to produce the high quality good.

Assume that the consumers do not observe quality before purchasing. It is clear that an equilibrium in which the monopolist sells and provides high quality cannot exist.

Suppose now that some consumers are informed about the quality of the product, say a fraction $\alpha$.

Observe first that if the informed consumers are buying then the uniformed consumer are buying as well.

The seller is better off selling to both segments of the market if:

$$p - c_1 \geq (1 - \alpha)(p - c_0)$$

or

$$\alpha p \geq c_1 - (1 - \alpha) c_0. \tag{1}$$

We can then make two observations:

- high quality is supplied only if the price is sufficiently high, "high price can signal high quality".
- a higher fraction, $\alpha$, of informed consumers favors efficiency as it prevents the monopolist from cutting quality.

- the informational externality favors government intervention as individuals only take private benefit and cost into account.

**Job Market Signaling**

**Basic Model**

Consider the following game first analyzed by Spence, 1973.

Nature chooses a worker's type (often interpreted as her productivity) $\theta \in \{1, 2\}$.

The worker has productivity $\theta$ with probability $p(\theta)$.

For notational convenience, we define

$$p \triangleq p(\theta = 2).$$

The worker chooses an educational level $e \in \mathbb{R}_+$. This choice is observable to two firms, but the firms do not observe directly the productivity $\theta$.

Two (or more) firms compete by offering wages $w_1, w_2$ to the worker given the observed level of education.

After seeing the wage offers, the worker chooses the firm with the more attractive wage offer. The worker's utility is

$$u(w, e, \theta) = w - \frac{e}{\theta}.$$

Since firms compete in wages, they offer the entire expected surplus in wages.

We observe also that the Spence-Mirrlees condition holds:

$$\frac{\partial^2 u(w, e, \theta)}{\partial e \partial \theta} = \frac{1}{\theta^2} > 0$$

and thus the worker's type $\theta$ and choice variable $e$ are complements.

An alternative terminology is that the worker's payoff is supermodular in $e$ and $\theta$.

## Pure Strategy

A pure strategy for the worker is a function

$$\widehat{e} : \{1, 2\} \rightarrow \mathbb{R}_+$$

A pure strategy for the firm is a function

$$w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$$

where $w(e)$ is the wage offered to a worker of educational level $e$.

## Perfect Bayesian Equilibrium

The novelty in signaling games is that the uninformed party gets a chance to update its prior belief on the basis of a signal sent by the informed party.

The updated prior belief is the posterior belief, depending on $e$ and denoted by $p(\theta \,| e)$.

The posterior belief is a mapping

$$\hat{p} : \mathbb{R}_+ \to [0, 1]$$

In sequential or extensive form games we required that the strategies are sequentially rational, or time consistent.

We now impose a similar consistency requirement on the posterior beliefs by imposing the Bayes' rule whenever possible.

Since $p(\theta | e)$ is a conditional probability, and hence a probability distribution on $\{1, 2\}$ for all values of $e$, it is required that:

$$\forall e \forall \theta,\ p(\theta | e) \geq 0, \text{ and } \sum_{\theta \in \{1,2\}} p(\theta | e) = 1. \tag{2}$$

Moreover, when the firm can apply Bayes law, it does so, or

$$\text{if } \exists\theta \text{ such that } e^*(\theta) = e, \text{ then } p(\theta\,|e\,) = \frac{p(\theta)}{\sum_{\{\theta'|e^*(\theta')=e\}} p(\theta')}. \quad (3)$$

We refer to educational choices which are selected by some worker-types in equilibrium as “on-the-equilibrium path”.

In other words, $e$ is on the equilibrium path if $\exists\theta$ such that $e^*(\theta) = e$.

Analogously, we say that $e$ is off the equilibrium path if there is no $\theta$ such that $e^*(\theta) = e$.

As before it will be sometimes easier to refer to

$$p(e) \triangleq p(\theta = 2\,|e\,)$$

and hence

$$p(1\,|e\,) = 1 - p(e)\,.$$

**Definition 1 (PBE)** *A pure strategy Perfect Bayesian Equilibrium is a set of strategies* $\{e^*(\theta), w^*(e)\}$ *and posterior beliefs* $p(e)$ *such that:*

1. $\forall e$, $\exists p(\theta \,|e\,)$, *s.th.* $p(\theta \,|e\,) \geq 0$, *and* $\sum_{\theta\in\{1,2\}} p(\theta \,|e\,) = 1$;
2. $\forall i$, $w_i^*(e) = \sum_\theta p(\theta \,|e\,)\,\theta$;
3. $\forall \theta$, $e^*(\theta) \in \arg\max \left\{w^*(e) - \frac{e}{\theta}\right\}$
4. *if* $\exists\theta$ *such that.* $e^*(\theta) = e$, *then:*

$$p(\theta \,|e\,) = \frac{p(\theta)}{\sum_{\{\theta' | e^*(\theta')=e\,\}} p\left(\theta'\right)}.$$

**Definition 2 (Separating PBE)** *A pure strategy PBE is a separating equilibrium if*

$$\theta \neq \theta' \Rightarrow e(\theta) \neq e\left(\theta'\right).$$

**Definition 3 (Pooling PBE)** *A pure strategy PBE is a pooling equilibrium if*

$$\forall \theta, \theta' \Rightarrow e(\theta) = e\left(\theta'\right).$$

**Theorem 4**

1. *A pooling equilibrium exists for all education levels* $e = e(1) = e(2) \in [0, p]$.

2. *A separating equilibrium exists for all* $e(1) = 0$ *and* $e(2) \in [1, 2]$.

**Proof.** (**??**) We first construct a pooling equilibrium. For a pooling equilibrium to exist it must satisfy the following incentive compatibility constraints

$$\forall e,\ 1 + p(e^*) - e^* \geq 1 + p(e) - e \tag{4}$$

and

$$\forall e,\ 1 + p(e^*) - \frac{e^*}{2} \geq 1 + p(e) - \frac{e}{2} \tag{5}$$

Consider first downward deviations, i.e. $e < e^*$, then (**??**) requires that

$$p(e^*) - p(e) \geq e^* - e. \tag{6}$$

Then consider upward deviations, i.e. $e > e^*$, then (**??**) requires that

$$p(e) - p(e^*) \leq \frac{1}{2}(e - e^*). \tag{7}$$

We can then ask for what education levels can both inequalities be satisfied. Clearly both inequalities are easiest to satisfy if

$$e < e^* \Rightarrow p(e) = 0$$
$$e > e^* \Rightarrow p(e) = 0,$$

which leaves us with

$$e < e^* \Rightarrow \quad p \geq e^* - e \tag{8}$$

$$e > e^* \Rightarrow \quad -p \leq \frac{1}{2}\left(e - e^*\right).$$

We may then rewrite the inequalities in (**??**) as

$$e < e^* \Rightarrow e^* \leq p + e \tag{9}$$

$$e > e^* \Rightarrow e^* \leq 2p + e$$

It is clear that the top inequality is the more demanding and that the top inequality is satisfied for all $e \geq 0$ if and only if $e^* \in [0, p]$. (**??**)

Consider then a separating equilibrium. It must satisfy the following incentive compatibility constraints

$$\forall e,\ 1 + p\left(e_1^*\right) - e_1^* \geq 1 + p\left(e\right) - e$$

and

$$\forall e,\ 1 + p\left(e_2^*\right) - \frac{e_2^*}{2} \geq 1 + p\left(e\right) - \frac{e}{2}. \tag{10}$$

As along the equilibrium path, the firms must apply Bayes law, we can rewrite the equations as

$$\forall e,\ 1 - e_1^* \geq 1 + p\left(e\right) - e$$

and

$$\forall e,\ 2 - \frac{e_2^*}{2} \geq 1 + p\left(e\right) - \frac{e}{2}.$$

Consider first the low productivity type. It must be that $e_1^* = 0$. This leaves us with

$$p\left(e\right) \leq e. \tag{11}$$

But if $e = e_2^*$ is supposed to be part of a separating equilibrium, then $p\left(e_2^*\right) = 1$.

Thus it follows further from (**??** ) that $e_2^* \geq 1$, for otherwise we would not satisfy $1 = p(e_2^*) \leq e_2^*$.

Finally, we want to determine an upper bound for $e_2^*$.

As we cannot require the high ability worker to produce too much effort otherwise he would mimic the lower ability, we can rewrite (**??**) to obtain:

$$1 - p(e) \geq \frac{1}{2}(e_2^* - e),$$

which is easiest to satisfy if

$$p(e) = 0,$$

and hence

$$\forall e,\ 2 + e \geq e_2^*$$

which implies that:

$$e_2^* \leq 2.$$

which completes the proof. ■

The theorem above only defines the range of educational choices which can be supported as an equilibrium but is not a complete equilibrium description as we have not specified the beliefs in detail.

The job market signaling model suggests a series of further questions and issues.

There were multiple equilibria, reducing the predictive ability of the model and we may look at different approaches to reduced the multiplicity:

- refined equilibrium notion
- different model: informed principal

The signal was modeled as a costly action. We may then ask for conditions:

- when do costly signals matter for Pareto improvements (or simply separation): Spence-Mirrlees single crossing conditions
- when do costless signals matter: cheap-talk games.

In the model, education could also be interpreted as an act of disclosure of information through the verification of the private information by a third party.

It may therefore be of some interest to analyze the possibilities of voluntary information disclosure.

## Equilibrium Domination

**Definition 5 (Equilibrium Domination)** *Given a $PBE$, the message $e$ is equilibrium-dominated for type $\theta$, if for all possible assessments $p\left(\theta\,|e\right)$ (or simply $p(e)$):*

$$w^*\left(e^*\left(\theta\right)\right) - \frac{e^*\left(\theta\right)}{\theta} > w\left(e\right) - \frac{e}{\theta}$$

*or equivalently*

$$1 + p\left(e^*\left(\theta\right)\right) - \frac{e^*\left(\theta\right)}{\theta} > 1 + p\left(e\right) - \frac{e}{\theta}$$

**Definition 6 (Intuitive Criterion)** *The posterior beliefs are said to satisfy the intuitive criterion if at any information set following any choice e that is off the equilibrium path and equilibrium dominated for type θ (and not dominated for some other type),*

$$p(\theta \mid e) = 0. \tag{12}$$

**Theorem 7 (Uniqueness)** *The unique Perfect Bayesian equilibrium outcome which satisfies the intuitive criterion is given by*

$$\{e_1^* = 0, e_2^* = 1,\ w^*(0) = 1, w^*(1) = 2\}. \tag{13}$$

*The beliefs are required to satisfy*

$$p^*(e) = \begin{cases} 0, & \textit{for } e = 0 \\ \in [0, e], & \textit{for } 0 < e < 1 \\ 1, & \textit{for } e \geq 1 \end{cases}$$

**Remark 8** *The equilibrium outcome is referred to as the least-cost separating equilibrium or Riley equilibrium.*

**Proof.** We first show that there can't be any pooling equilibria satisfying the intuitive criterion, and then proceed to show that only one of the separating equilibria survives the test of the intuitive criterion.

Suppose $0 \leq e^* \leq p$.

Consider $e \in (e^* + (1-p), e^* + 2(1-p))$. Any message $e$ in the interval is equilibrium dominated for the low productivity worker as

$$1 + p - e^* > 2 - e,$$

but any message in the interval is not equilibrium dominated for the high productivity worker, as

$$1 + p - \frac{e^*}{2} < 2 - \frac{e}{2}$$

and thus for $e = e^* + (1-p)$, we have

$$p < 1 - \frac{(1-p)}{2} \Leftrightarrow \frac{1}{2}p < \frac{1}{2},$$

which certainly holds.

Thus if $(e, p(e))$ satisfies the intuitive criterion, we must have $p(e) = 1$ for $e \in (e^* + (1-p), e^* + 2(1-p))$ , but then pooling is not an equilibrium anymore as the high productivity worker has a profitable deviation with any $e \in (e^* + (1-p), e^* + 2(1-p))$.

Consider now the separating equilibria. For $e_2^* > 1$, any $e \in (1, e_2^*)$ is equilibrium dominated for the low ability worker as

$$1 > 2 - e,$$

but is not equilibrium dominated for the high ability worker, as

$$2 - \frac{e_2^*}{2} < 2 - \frac{e}{2}.$$

It follows that $p(e) = 1$ for all $e \in (1, e_2^*)$.

But then $e_2^* > 1$, cannot be supported as an equilibrium as the high ability worker has a profitable deviation by lowering his educational level to some $e \in (1, e_2^*)$, and still receive his full productivity in terms of wages.

It remains $e_2^* = 1$ as the unique PBE satisfying the intuitive criterion.

■

**Criticism.**

i) Suppose $p \to 1$.

ii) Stglitz Critique: The reason why the pooling equilibria do not satisfy the intuitive criterion is the following.

In the best pooling equilibrium, both types choose $e = 0$.

There are nevertheless educational levels $e'$ such that if the firms believe that the workers are of high productivity, $e'$ yields a higher payoff to those with $\theta = 2$, but a strictly lower payoff than the pooling equilibrium payoff for those with $\theta = 1$.

Hence if such an educational choice is ever observed, the intuitive criterion requires the firms to conclude that the worker indeed has $\theta = 2$. But then $e = 0$ is no longer optimal for those with $\theta = 2$.

Stglitz Critique asks the following question: If this is all understood amongst the players and if the firms still observe $e = 0$, shouldn't they conclude that the worker is of type $\theta = 1$.

In this case, the worker's payoff is much lower than in the original equilibrium, and it may indeed be in the workers' best interest to play $e'$ just to avoid this conclusion.

But then there are no grounds for requiring that the posterior belief after seeing $e = e'$ should be different from the prior. Hence the pooling equilibrium should still be considered an equilibrium.

## Maskin and Tirole's informed principal problem

To elaborate on the first criticism, assume that $p$, the prior probability that a worker is of type $\theta = 2$, is arbitrarily large, $(p \longrightarrow 1)$. In that case, it seems a rather high cost to pay to incur an education cost of

$$c(e = 1) = \frac{1}{2}$$

just to be able to raise the wage by an extremely small amount $\Delta w = 2 - (1 + p) \longrightarrow 0$ as $p \rightarrow 1$.

Indeed, in that case the pooling equilibrium where no education costs are incurred seems a more plausible outcome.

This particular case should serve as a useful warning not to rely too blindly on selection criteria (such as Cho-Kreps' intuitive criterion) to single out particular PBEs.

Interestingly, much of the problem of multiplicity of PBEs disappears when the timing of the game is changed to letting agent and principal sign a contract before the choice of signal.

This is one important lesson to be drawn from Maskin and Tirole, 1992.

To see this, consider the model of education as specified above and invert the stages of contracting and education choice.

That is, now the worker signs a contract with his/her employer before undertaking education.

This contract then specifies a wage schedule contingent on the level of education chosen by the worker after signing the contract.

Let $\{w(e)\}$ denote the contingent wage schedule specified by the contract.

Consider the problem for the high productivity worker.

Suppose he would like to make an offer by which he can separate himself from a low ability worker

$$\max_{\{e,w(e)\}} \left\{ w(e) - \frac{e}{2} \right\}$$

subject to

$$w(e_1) - e_1 \geq w(e_2) - e_2 \qquad (IC_1)$$

and

$$w(e_2) - \frac{e_2}{2} \geq w(e_1) - \frac{e_1}{2} \qquad (IC_2)$$

and

$$\theta_1 - w(e_1) \geq 0 \qquad (IR_1)$$

$$\theta_2 - w(e_2) \geq 0 \qquad (IR_2)$$

Thus to make incentive compatibility as easy as possible he suggests $e_1 = 0$ and $w(e_1 = 0) = 1$. As $w(e_2) = 2$, it follows that after setting $e_2 = 1$, he indeed maximizes his payoff.

Suppose instead he would like to offer a pooling contract. Then he would suggest

$$\max_{e,w} \left\{ w - \frac{e}{2} \right\}$$

$$1 + p - w \geq 0 \qquad (IR_1)$$

which would yield $w = 1 + p$ for all $e$.

This suggest that there are two different cases to consider:

1. $1 + p \leq 2 - \frac{1}{2}$: a high productivity worker is better off in the "least cost" separating equilibrium than in the efficient pooling equilibrium.
2. $1 + p > 2 - \frac{1}{2}$: a high productivity worker is better off in the efficient pooling equilibrium.

It is easy to verify that in the case where $p \leq \frac{1}{2}$, the high productivity worker cannot do better than offering the separating contract, nor can the low productivity worker. More precisely, the high productivity worker strictly prefers this contract over any contract resulting in pooling or any contract with more costly separation. As for the low productivity worker, he has everything to lose by offering another contract which would identify himself.

In the alternative case where $p > \frac{1}{2}$ , the unique equilibrium contract is the one where:

$$w^*(e) = 1 + p \text{ for all } e \geq 0.$$

Again, if the firm accepts this contract, both types of workers choose an education level of zero.

Thus, on average the firm breaks even by accepting this contract, provided that it is as (or more) likely to originate from a high productivity worker than a low productivity worker.

Now, a high productivity worker strictly prefers to offer this contract over any other separating contract.

Similarly, a low productivity worker has everything to lose from offering another contract and thus identifying himself.

Thus, in this case again this is the unique contract offer made by the workers.

**Spence-Mirrlees Single Crossing Condition**

**Separating Condition**

Consider now a general model with a compact interval of types:

$$\Theta \subseteq \mathbb{R}_+$$

and an arbitrary compact interval of signals

$$E \subseteq \mathbb{R}_+$$

with a general quasilinear utility function

$$u(t, e, \theta) = t + v(e, \theta)$$

where we recall that the utility function used in Spence model was given by:

$$u(t, e, \theta) = t - \frac{e}{\theta}$$

We now want ask when is it possible in general to sustain a separating equilibrium such that

$$\theta \neq \theta' \Rightarrow e \neq e'$$

Suppose we can support a separating equilibrium with strict incentives for choosing the equilibrium level of education, and furthermore that the equilibrium level of education is strictly increasing in $\theta$. Then we must be able to satisfy for all $\theta' > \theta$ and for the corresponding equilibrium education levels $e' > e:$

$$t + v(e,\theta) \geq t' + v(e',\theta) \Leftrightarrow t - t' \geq v(e',\theta) - v(e,\theta) \quad (14)$$

and

$$t' + v\left(e',\theta'\right) \geq t + v\left(e,\theta'\right) \Leftrightarrow t - t' \leq v\left(e',\theta'\right) - v\left(e,\theta'\right), \quad (15)$$

$$v(e',\theta) - v(e,\theta) \leq v\left(e',\theta'\right) - v\left(e,\theta'\right). \quad (16)$$

Hence we see that a necessary condition for separation is that

$$v\left(e',\theta'\right) + v(e,\theta) \geq v(e',\theta) + v\left(e,\theta'\right) \text{ for all } \theta > \theta' \text{ and } e > e'. \quad (17)$$

Functions that satisfy **??** for all $e, \theta$ are said to be strictly supermodular in $(e,\theta)$. In the differentiable case, supermodularity coincides with a strictly positive cross partial derivative. If we want to get a separating equilibrium with $e(\theta)$ that is strictly decreasing in $\theta$, the condition identified by **??** is called submodularity (and corresponds to negative cross partials).

Consider next what properties $t$ must satisfy to be compatible with separating equilibria. As long as $\frac{\partial v(e,\theta)}{\partial e} < 0$, a necessary condition for getting separating equilibria is that $\frac{dt(e)}{de} > 0$. In the discussion to follow, I concentrate on the case where $\frac{de(\theta)}{d\theta} > 0$ and thus $v(e, \theta)$ is supermodular. My claim is that for all $t(e)$ with $\frac{dt(e)}{de} > 0$, a separating equilibrium (usually) exists. The proof is by construction.

Consider the agent's problem:

$$\max_e \; t(e) + v(e, \theta).$$

The first order condition is given by

$$\frac{dt(e)}{de} + \frac{\partial v(e,\theta)}{\partial e} = 0.$$

This equation defines implicitly the optimal $e$ for type $\theta$. To see that all agents choose different levels of $e$, differentiate the first order condition with respect to $\theta$ and $e$ to get:

$$-\frac{\frac{\partial^2 v(e,\theta)}{\partial e \partial \theta}}{\left[\frac{d^2 t(e)}{de^2} + \frac{\partial^2 v(e,\theta)}{\partial e^2}\right]} = \frac{de}{d\theta}. \tag{18}$$

The denominator is nonpositive by the second order condition to the maximization problem. If it is strictly negative, then the conditions for the existence of a separating equilibrium are satisfied. Denote the lowest type and education level by $\underline{\theta}$ and $\underline{e}$ respectively. Let $e\left(\underline{\theta}\right) = \underline{e}$, and define the education levels for higher types through the differential equation that is defined in **??**. This completes the construction of the separating equilibrium.

## Cheap Talk

Another famous type of games is closely related to signaling games. In Cheap Talk games, there is an informed party, often called the expert, and an uninformed party, called the decision maker. At the start of the game, the expert can announce her information at no cost. After the announcement, the decision maker chooses an action that has payoff consequences for both the expert and the decision maker.

Formally, we have then a two player game where the first mover, i.e. the expert, has private information or type $t \in T$. Her strategy is to report a (possibly mixed) message $m$ conditional on type, i.e.

$$r : T \to \Delta(M)$$

The decision maker observes the message $m$ (but not type $t$) and chooses an action (again this is possibly a random action):

$$a : M \to \Delta(A)$$

The payoffs of both parties depend on the type and the action, but not on the signal itself. This is the distinguishing feature relative to other signaling models. We write the expert's payoff as $v(t, a)$ and the decision maker's payoff is $u(t, a)$. Finally, the description of the game is completed by defining a prior probability distribution $\mu$ on $T$.

The key to understanding the model lies in the analysis of the decision maker's problem. After observing a message $m$, the decision maker updates her beliefs on the type of the expert and then chooses optimally. If the observed message is on the equilibrium path, then beliefs are updated according to Bayes' rule. When using perfect Bayesian equilibrium as the solution concept, there is a lot of freedom in specifying beliefs for information sets that are off the equilibrium path.

As always, perfect Bayesian equilibrium is a strategy for the expert, a belief for the decision maker for each possible message observation and an optimal action choice for the decision maker given the belief.

The first observation on Cheap Talk games is that these games have always a pooling equilibrium where no information is transmitted. This is simply given by the strategies

$$s(t) = m' \text{ for all } t \text{ and for some } m'.$$

The posterior of the decision maker is the prior $\mu$ for all $m \in M$ and the strategy of the decision maker is

$$a(m) = a' \text{ where } a' \in argmax_a E_\mu u(t, a)$$

The second observation is that if

$$argmax_a u(t, a) = argmax_a v(t, a) \text{ for all } t,$$

and $M$ has at least the same cardinality as $T$, then the game has a separating equilibrium.

The more interesting question is whether the game has equilibria where some information is transmitted when there is a genuine conflict of interest between the expert and the decision maker, i.e. when condition **??** does not hold.

The model by Crawford and Sobel (1982) showed in a general setting that Cheap Talk games do in general possess partially revealing equilibria (alternatively these equilibria could be called partially pooling) and thus some, but not all information can be transmitted.

The last decade has seen an explosion of interest in cheap talk games.

A major reason for this has been the shift in political science towards more formal models.

The relationship between informed parties (lobby groups) and uninformed decision makers (politicians) has a large number of potential applications there.