FDPE Microeconomic Theory
Fall 2009
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Problem Set 5 (Due Nov 4)

1. Consider a risk-neutral agent who owns a call option on an underlying asset whose value at the excercise date is $\widetilde{x}$. The call option gives to the agent the option to purchase the asset at a prespecified price p, called excercise price. Show that the agent likes an increase in risk in the value of the asset. How would your answer change if the agent owns $n > 1$ call options with exercise prices $p_1, ..., p_n$?

2. An investor has initial wealth w_0 that she must divide between a safe and a risky asset. Denote the fraction of initial wealth invested in the risky asset by α. There are two possible states, H and L, for the economy. Assume that the economy will be in state H with probability p and in L with probability $(1-p)$ .The risky asset pays a gross return R^H if the state of the economy is H and a gross return 0 if the state is L. I.e. in state H, an investment of αw_1 in the risky asset results in a return $\alpha w_1 R^H$ and in state L , the return is 0. A unit of the safe asset pays R in both states.

   (a) For an arbitrary investment portfolio $\{\alpha, 1-\alpha\}$, determine the final wealth of the investor.

   (b) Assume the investor has a Bernoulli utility function $u(w)$ where w denotes the final wealth. Assume that $u'(w) > 0$ and $u''(w) < 0$. Write down the expected utility resulting from an arbitrary portfolio and characterize the first order conditions for the optimal portfolio. Argue also that second order conditions are also satisfied at the point that satisfies the first order conditions.

   (c) Consider now a third asset that pays 0 in H and R^L per unit invested in L. For what values of R^H, R^L, R will the investor invest nothing in the safe asset? Is it ever optimal to invest a stricly positive fraction of wealth in all three assets? (hint: construct a "safe" asset by combining risky assets at a suitable ratio, and compare its return with the original safe asset).

3. (MWG 6.C.5) Consider a consumer with utility function $u(\cdot) : \mathbb{R}_+^L \rightarrow \mathbb{R}$ defined over bundles of L goods, just like in lecture 3.

   (a) Argue that concavity of $u(\cdot)$ can be interpreted as the decision maker exhibiting risk aversion with respect to lotteries whose outcomes are bundles of the L commodities.

   (b) Suppose that a Bernoulli utility function $\widetilde{u}(w)$ for wealth is derived from the utility maximization problem defined over bundles of consumption for each given wealth level w, keeping the price vector p for commodities fixed. Show that, if the utility function u for the commodities exhibits risk aversion, then so does the derived Bernoulli utility function for wealth. Interpret.

   (c) Can you find an example that shows that the converse of (b) does not necessarily hold? (take $L = 2$, and look for a nonconcave function u such that the derived Bernoulli utility function on wealth exhibits risk aversion)

4. Consider an economy with one representative agent and two dates, $t = 0, 1$. The economy has an exogenous consumption process in the following sense: in the absence of savings, period $t = 1$ consumption, $\tilde{c}_1$, is distributed according to some cumulative distribution function $F(c)$. Thus, $E\{\tilde{c}_1\} = \int c dF(c)$. Period $t = 0$ consumption c_0 is given. The agent has separate attitudes towards time and risk, so we seek to formulate preferences such that the two can disentangled. The utility over time is given by

$$U(c_0, \tilde{c}_1) = u(c_0) + \beta u(c(v, F))$$

   where $\beta \in (0, 1)$ is the discount factor, u is a weakly concave function, and $c(v, F)$ is the certainty equivalent consumption for period $t =$

1 using another weakly concave function v. That is, $v(c(v, F)) = E\{v(\tilde{c}_1)\}$. Note first that if $u = v$ we have the usual time-separable expected-utility objective. Second, given $c(v, F)$, all uncertainty has been removed from calculations using $U(c_0, \tilde{c}_1)$, so the concavity of u relates to consumption smoothing only. Third, concavity of v measures risk aversion only.

(a) Suppose the agent can invest and sacrify consumption at $t = 0$ to achieve a sure benefit $(1 + r)$ per unit invested at $t = 1$. Now, if you can find a rate of return r that makes the agent just indifferent between investing and not investing, given c_0 and the expected $\tilde{c}_1$, you have found the socially efficient discount rate for this economy. Do this and discuss how it depends on consumption smoothing and risk aversion.

(b) Consider now the effect of increasing uncertainty on the discount rate. To obtain a benchmark compute the socially efficient discount rate under the assumption that $E\{\tilde{c}_1\}$ is the period $t = 1$ consumption for sure. Denote this by r^c. Then, calculate the true socially efficient discount rate under the assumption that $\tilde{c}_1$ is uncertain. Does the uncertainty reduce the discount rate? Assume time separable preferences, that is, $v = u$.

(c) The same problem as above, but assume now $v \neq u$. Show that the socially efficient discount rate falls with uncertainty about future income levels if the agent is decreasingly absolute risk averse.

5. In models of oligopolistic competition, it is typical that the profit of a firm lagging behind the leader in the industry in terms of the quality of its product has a profit function that is first convex and then concave in any improvements to its own quality. R&D investments within a firm result normally in random improvements in the quality A possible way of modeling the R&D activity is by considering the choice of various types of projects, i.e. various distributions over quality improvements. With this as a motivation, consider the following model. The profit of

the firm is given by function $\pi(x)$ defined for $x \geq 0$, and there is an x_0 such that $\pi''(x) \geq 0$ for all $x \leq x_0$, and $\pi''(x) \leq 0$ for all $x \geq x_0$, and furthermore $\pi'(x) > 0$ for all x and $\lim_{x \to \infty} \pi'(x) = 0$. The firm maximizes its expected profit by making a choice amongst different random distributions for x on $\mathbb{R}_+$.

(a) Assume that the firm can choose any probability distribution for quality improvement x under constraint that the expected value of the quality improvement is fixed. That is, the firm wants to choose the distribution for x in order to maximize $\mathbb{E}[\pi(x)]$ subject to $\mathbb{E}x = \mu$. Argue that the distribution that maximizes expected profit has at most two points in its support, and characterize the optimal distributions.

(b) Suppose that there is a cost of increasing μ. More specifically, let $c(\mu)$ denote the cost, and assume that $c, c', c'' \geq 0$. Find the optimal distribution (i.e. find optimal μ and the corresponding distribution that maximizes $\mathbb{E}[\pi(x)] - c(\mu)$ s.t. $\mathbb{E}x = \mu$)