

Microeconomic Theory I

Pauli Murto

Finnish Doctoral Programme in Economics

Autumn 2009

- In this lecture we continue with General Equilibrium
- First, we discuss existence and uniqueness issues of equilibrium
- ▣ Then we move to the topic of general equilibrium under uncertainty
- We discuss contingent commodities and Arrow-Debreu equilibrium
- This leads naturally to the analysis of asset markets
- The main material for this lecture is MWG 17 A-C, 19 A-E

Exchange economies and excess demand function

- For most of this lecture, for simplicity, we concentrate on pure exchange economies
- Formally, the only possible technology is then free disposal: $Y = 1$ with $Y_1 = -\mathbb{R}_+^L$
- As before, there are $I > 0$ consumers and $L > 0$ commodities
- Consumption set for each consumer is $X_i = \mathbb{R}_+^L$
- Consumer i has preferences \succsim_i defined on X_i , which we assume to be continuous, strictly convex, and locally nonsatiated (sometimes, strongly monotonic)
- ▣ Endowments are given for each individual by ω_i , where $\sum_i \omega_i \gg 0$

- An important concept is the excess demand function:

Definition

The excess demand function of consumer i is

$$z_i(p) := x_i(p, p \cdot \omega_i) - \omega_i,$$

where $x_i(p, p \cdot \omega_i)$ is i 's Walrasian demand function. The aggregate excess demand function is

$$z(p) = \sum_{i=1}^I z_i(p).$$

- Assuming locally nonsatiated preferences, the following properties of $z(p)$ follow easily from the properties of Walrasian demand function. These hold for strictly positive price vectors $p \gg 0$:
 - $z(\cdot)$ is continuous
 - $z(\cdot)$ is homogenous of degree zero
 - $p \cdot z(\cdot) = 0$ for all p (Walras' law)
- The Walrasian equilibrium can then be expressed succinctly using excess demand function:

Theorem

In a pure exchange economy with continuous, strictly convex, and locally nonsatiated preferences, $p \geq 0$ is a Walrasian equilibrium price vector if and only if $z(p) \leq 0$.

- Equilibrium allocations are simply Walrasian demands with the equilibrium price
- Note we also must have $z_i(p) = 0$ if $p_i > 0$ (this follows from Walras' law)
- If we adopt the stronger assumption that preferences are strongly monotone, then the equilibrium price must be strictly positive for each commodity, that is, $p \gg 0$ (otherwise consumers would demand infinite amounts of the free good)
- It then follows that with strongly monotonous preferences, equilibrium is simply characterized by the property $z(p) = 0$.

Existence of equilibrium in exchange economies

- Let us state a version of existence theorem valid for all positive price vectors:

Theorem

If $z(p)$ is a function defined for all $p \in \mathbb{R}_+^L$, is continuous, homogenous of degree zero, and satisfies Walras' law, then there is a price vector p^* such that $z(p^*) \leq 0$.

- This version is easy to prove (uses Brouwer's fixed point theorem, and will be discussed in the class), but the requirement of continuity is problematic: it will not hold at $p_l = 0$ with strongly monotonous preferences, because with such preferences demand of l goes to infinity at $p_l \rightarrow 0$
- There is a more applicable version of the existence theorem: Proposition 17.C.1 in MWG
- Note that another assumption in the theorem is the requirement that demand is a function rather than correspondence. This is true with convex preferences (or with a large number of consumers - see Section 17.1 in MWG)

Uniqueness of equilibrium (just brief remarks)

- Uniqueness is important in terms of the predictive power of the model
- Unfortunately, uniqueness in general equilibrium models can only be guaranteed under special circumstances
- One such case is: if all pairs of goods are gross substitutes of each other at all prices, then there is a unique equilibrium
- Without global uniqueness, we can still hope for local uniqueness (that is, each equilibrium price vector is unique in its neighbourhood)
- Local uniqueness holds generally in general equilibrium models (it is a generic property)
- More about uniqueness: MWG sections 17.D, 17.F

General equilibrium under uncertainty

- We represent uncertainty using states
- Assume that there are S distinct states, $s = 1, \dots, S$
- As before, there are L physical commodities ($l = 1, \dots, L$)
- We denote by $x = (x_{11}, \dots, x_{L1}, \dots, x_{1S}, \dots, x_{LS}) \in \mathbb{R}^{LS}$ a *contingent commodity vector*
- Here x_{ls} represents the amount of physical commodity l obtained if state s is realized (a negative x_{ls} means obligation to deliver good l in state s)
- Endowments are also defined contingent on state: $\omega_i = (\omega_{11i}, \dots, \omega_{L1i}, \dots, \omega_{1Si}, \dots, \omega_{LSi}) \in \mathbb{R}^{LS}$
- We represent preferences of i as a rational preference relation \succsim_i defined on consumption set $X_i \subset \mathbb{R}^{LS}$
- Technologies are described by a state-contingent production possibilities set $Y_j \subset \mathbb{R}^{LS}$
- For simplicity, assume that ownership shares are state independent: $\theta_{ji} \geq 0, \sum_j \theta_{ji} = 1$

Discussion:

- Note that preferences are defined over contingent commodity vectors
- We can easily accommodate expected utility formulation. That is, let

$$x_i \succsim_i x'_i \iff \sum_s \pi_{sj} u_{sj}(x_{1s1}, \dots, x_{Ls1}) \geq \sum_s \pi_{sj} u_{sj}(x'_{1s1}, \dots, x'_{Ls1}),$$

where π_{sj} is the probability (assigned by i) of state s , and u_{sj} is i 's Bernoulli state-dependent utility function

- Convexity of preferences relates to risk aversion: if there is an expected utility representation, \succsim_i is convex when Bernoulli utility functions of i are concave
- Note that preferences are "ex-ante" preferences: they rank contingent commodities before uncertainty is resolved
- Decisions that are made before or after resolution of uncertainty can be distinguished by defining Y_j appropriately (similarly for consumption decisions by defining X_i)

Arrow-Debreu Model

- Arrow-Debreu model refers to a world, where all possible contingent commodities have well functioning markets
- These markets clear before the resolution of uncertainty
- Price p_{ls} is the price of an entitlement to one unit of commodity l in case state s realizes
- Seller of such an entitlement must deliver a unit of l if s realizes
- Existence of all these markets is of course a strong assumption
- The model is actually just a particular specification of the general equilibrium model that we have been analyzing

- Walrasian equilibrium in this context is called an Arrow-Debreu equilibrium:

Definition

An allocation

$(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*) \in X_1 \times \dots \times X_I \times Y_1 \times \dots \times Y_J \subset \mathbb{R}^{LS(I+J)}$ and a price vector $p = (p_{11}, \dots, p_{LS}) \in \mathbb{R}^{LS}$ constitute an Arrow-Debreu equilibrium if:

- For every j , y_j^* maximizes profits in Y_j ; that is

$$p \cdot y_j \leq p \cdot y_j^* \text{ for all } y_j \in Y_j.$$

- For every i , x_i^* is maximal for \succsim_i in the budget set

$$\left\{ x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^* \right\}.$$



$$\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$$

- Note that this is in no way different to our earlier definition of Walrasian equilibrium
- In particular, welfare theorems and existence/uniqueness issues discussed earlier apply to this context
- Since all commodities are traded ex-ante contingent on all realizations of s , there is no scope for ex-post trade. Ex-ante Pareto-optimality implies ex-post Pareto-optimality.
- But clearly the existence of markets for all contingent commodities is very strong assumption indeed
- Fortunately, we can do with less
- In fact, we'll see that contingent markets for one physical good for all states is enough
- Intuition: one commodity is enough to transfer wealth across states to reach the same outcome (with appropriate ex-post trade after the realization of state)

Sequential Trade

- So, let us consider sequential trade, where there is contingent trade on commodity 1 at date $t = 0$, and ex-post trade on all commodities at date $t = 1$ (after uncertainty has resolved)
- For simplicity, assume pure exchange
- Denote the prices of the contingent commodities traded at $t = 0$ by $q = (q_1, \dots, q_S) \in \mathbb{R}^S$
- Denote the prices of all commodities at $t = 1$ (spot prices) by $p = (p_1, \dots, p_S) \in \mathbb{R}^{LS}$
- Denote by $(z_{1i}, \dots, z_{Si}) \in \mathbb{R}^S$ the trades of i of the contingent commodity
- Denote by $(x_{1i}, \dots, x_{Si}) \in \mathbb{R}^{LS}$ the spot market consumption plan of i
- Assume that consumers have correct expectations of spot prices when they trade on forward markets

- Denoting by $U_i(\cdot)$ the utility for \succsim_i , consumer i 's problem is:

$$\text{Max}_{(z_{1i}, \dots, z_{Si}), (x_{1i}, \dots, x_{Si})} U_i(x_{1i}, \dots, x_{Si})$$

$$\text{s.t. (i) } \sum_s q_s z_{si} \leq 0$$

$$\text{(ii) } p_s \cdot x_{si} \leq p_s \cdot \omega_{si} + p_{1s} z_{si} \text{ for every } s$$

- Assuming consumers' expectations of spot prices are rational, the equilibrium can be stated as:

Definition

A collection formed by $q = (q_1, \dots, q_S) \in \mathbb{R}^S$, $p = (p_1, \dots, p_S) \in \mathbb{R}^{LS}$, $z_i^* = (z_{1i}^*, \dots, z_{Si}^*) \in \mathbb{R}^S$ for all $i = 1, \dots, I$, and $x_i^* = (x_{1i}^*, \dots, x_{Si}^*) \in \mathbb{R}^{LS}$ for all $i = 1, \dots, I$, constitutes a Radner equilibrium if:

- For every i , consumption plans z_i^* , x_i^* solve the problem of the previous slide
- $\sum_i z_{Si}^* \leq 0$ and $\sum_i x_{Si}^* \leq \sum_i \omega_{Si}$ for every s

- Note that trade takes place sequentially, but the consumers take into account period $t = 1$ trades in optimizing period $t = 0$ trades
- It is natural to normalize $p_{1s} = 1$ for each s , so a unit of the s -contingent commodity pays off 1 dollar in state s

- Importantly, in the equilibrium defined above, allocations are identical to the Arrow-Debreu equilibrium:

Theorem

(i) If $x^* \in \mathbb{R}^{LSI}$ and $p = (p_1, \dots, p_S) \in \mathbb{R}_{++}^{LS}$ constitute an Arrow-Debreu equilibrium, then there are $q \in \mathbb{R}_{++}^S$ and $z^* \in \mathbb{R}^{SI}$ such that (q, p, z^*, x^*) constitute a Radner equilibrium

(ii) If $(q, p, z^*, x^*) \in \mathbb{R}_{++}^S \times \mathbb{R}_{++}^{LS} \times \mathbb{R}^{SI} \times \mathbb{R}^{LSI}$ constitute a Radner equilibrium, then there are multipliers $(\mu_1, \dots, \mu_S) \in \mathbb{R}_{++}^S$ such that x^* and $(\mu_1 p_1, \dots, \mu_S p_S) \in \mathbb{R}_{++}^{LS}$ constitute an Arrow-Debreu equilibrium.

- Idea of proof: budget sets are identical in the two settings
- Multiplier μ_s is to be interpreted as the value, at $t = 0$, of a dollar at $t = 1$ in state s .

- The role of contingent commodities above is to transfer wealth ex-ante across future states of world
- In reality, assets do not look quite like the ones defined above
- So, the question we ask here is, can we do the same with other kind of assets?
- The answer will be yes, if the asset markets are complete (in a sense to be defined)
- We continue to assume: two dates $t = 0$ and $t = 1$, where consumption takes place only at date $t = 1$
- This could be generalized with no conceptual difficulties

Definition

A unit of asset, or security, is a title to receive an amount r_s of good 1 at date $t = 1$ if state s occurs. An asset is characterized by its return vector $r = (r_1, \dots, r_S) \in \mathbb{R}^S$.

- Examples of assets:
 - $r = (1, \dots, 1)$ is a commodity future
 - $r = (0, \dots, 0, 1, 0, \dots, 0)$ is contingent commodity discussed earlier (an Arrow security)
 - $r = (\max\{0, r_1 - c\}, \dots, \max\{0, r_S - c\})$ is an option, where $r = (r_1, \dots, r_S)$ is a primary asset and c is the strike price (options are derivative assets, because they derive their value from that of another asset)
- We can summarize the asset structure in a $S \times K$ return matrix R :

$$R = \begin{bmatrix} r_{11}, \dots, r_{1K} \\ \dots \\ r_{S1}, \dots, r_{SK} \end{bmatrix}$$

- Return of portfolio z is thus Rz

- Given K assets traded at prices $q = (q_1, \dots, q_K)$ at $t = 0$, the equilibrium is:

Definition

A collection formed by $q = (q_1, \dots, q_K) \in \mathbb{R}^K$, $p = (p_1, \dots, p_S) \in \mathbb{R}^{LS}$, $z_i^* = (z_{1i}^*, \dots, z_{Ki}^*) \in \mathbb{R}^K$ for all $i = 1, \dots, I$, and $x_i^* = (x_{1i}^*, \dots, x_{Si}^*) \in \mathbb{R}^{LS}$ for all $i = 1, \dots, I$, constitutes a Radner equilibrium if:

- For every i , consumption plans z_i^* , x_i^* solve

$$\text{Max}_{(z_{1i}, \dots, z_{Ki}), (x_{1i}, \dots, x_{Si})} U_i(x_{1i}, \dots, x_{Si})$$

- s.t. (i) $\sum_k q_k z_{ki} \leq 0$
 (ii) $p_s \cdot x_{si} \leq p_s \cdot \omega_{si} + \sum_k p_{1s} z_{ki} r_{sk}$ for every s

- $\sum_i z_{ki}^* \leq 0$ and $\sum_i x_{si}^* \leq \sum_i \omega_{si}$ for every k and s

- Return matrix restricts the set of possible asset price vectors. This result is an expression of a very important principle: in equilibrium, asset prices must be arbitrage-free.

Theorem

Assume that every return vector is nonnegative and nonzero. Then, for every price vector $q \in \mathbb{R}^K$ arising in a Radner equilibrium, we can find multipliers $\mu = (\mu_1, \dots, \mu_S) \geq 0$, such that $q_k = \sum_s \mu_s r_{sk}$ for all k . In matrix notation, $q^T = \mu \cdot R$.

- Two versions of proof given in MWG, section 19.E

- Let us now give the key definition:

Definition

An asset structure with an $S \times K$ return matrix R is complete if $\text{rank}(R) = S$, that is, if there is a subset of S assets with linearly independent returns.

- With S linearly independent returns, one can obtain any return profile by choosing portfolio appropriately
- Thus, with a complete asset structure, agents are again unrestricted in their wealth transfers across states
- It is therefore not surprising, that we again end up in the same allocation as in Arrow-Debreu equilibrium:

Theorem

Suppose that the asset structure is complete. Then:
 (i) If $x^* \in \mathbb{R}^{LS}$ and $p = (p_1, \dots, p_S) \in \mathbb{R}_{++}^{LS}$ constitute an Arrow-Debreu equilibrium, then there are asset prices $q \in \mathbb{R}_{++}^K$ and portfolio plans $z^* \in \mathbb{R}^{KI}$ such that (q, p, z^*, x^*) constitute a Radner equilibrium
 (ii) If $(q, p, z^*, x^*) \in \mathbb{R}_{++}^K \times \mathbb{R}_{++}^{LS} \times \mathbb{R}^{KI} \times \mathbb{R}^{LS}$ constitute a Radner equilibrium, then there are multipliers $(\mu_1, \dots, \mu_S) \in \mathbb{R}_{++}^S$ such that x^* and $(\mu_1 p_1, \dots, \mu_S p_S) \in \mathbb{R}_{++}^{LS}$ constitute an Arrow-Debreu equilibrium.

- Idea of proof as before (see MWG)
- Again, multiplier μ_s is the value, at $t = 0$, of a dollar at $t = 1$ in state s

- An important principle is that of *pricing by arbitrage*.
- To see how it works, assume there is a complete asset structure
- Then there must be some S assets with linearly independent returns
- Use their prices q_k to deduce the state multipliers $\mu = (\mu_1, \dots, \mu_S)$
(solve μ from S linear equations $q_k = \sum_s \mu_s r_{sk}$, $k = 1, \dots, S$)
- Once you have the state multipliers, price of any other asset k' with return vector $r_{k'}$ must be $q_{k'} = \sum_s \mu_s r_{sk'}$
- Otherwise there would be an arbitrage opportunity. Can you see why?
It is a good exercise to work this out in more detail
- Note that state multipliers do not depend directly on state probabilities
- This issue is at the heart of a large body of finance theory
- With incomplete markets, things are much more complicated