Microeconomic Theory I

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Lecture 8: General Equilibrium

- In this lecture we continue with General Equilibrium
- First, we discuss existence and uniqueness issues of equilibrium
- Then we move to the topic of general equilibrium under uncertainty
- We discuss contingent commodities and Arrow-Debreau equilibrium
- This leads naturally to the analysis of asset markets
- The main material for this lecture is MWG 17 A-C, 19 A-E

Exchange economies and excess demand function

- For most of this lecture, for simplicity, we concentrate on pure exchange economies
- Formally, the only possible technology is then free dispolsal: $Y = 1$ with $Y_1 = -\mathbb{R}^L_+$
- As before, there are $I > 0$ consumers and $L > 0$ commodities
- Consumption set for each consumer is $X_i = \mathbb{R}^L_+$
- Consumer i has preferences $\succsim_i$ defined on X_i, which we assume to be continuous, strictly convex, and locally nonsatiated (sometimes, strongly monotonic)
- Endowments are given for each individual by ω_i, where $\sum_i \omega_i \gg 0$

- An important conecept is the excess demand function:

Definition

The excess demand function of consumer i is

$$z_i(p) := x_i(p, p \cdot \omega_i) - \omega_i,$$

where $x_i(p, p \cdot \omega_i)$ is i's Walrasian demand function. The aggregate excess demand function is

$$z(p) = \sum_{i=1}^{I} z_i(p).$$

- Assuming locally nonsatiated preferences, the following properties of $z(p)$ follow easily from the properties of Walrasian demand function. These hold for strictly positive price vectors $p \gg 0$:
  - $z(\cdot)$ is continuous
  - $z(\cdot)$ is homogenous of degree zero
  - $p \cdot z(\cdot) = 0$ for all p (Walras' law)
- The Walrasian equilibrium can then be expressed succinctly using excess demand function:

Theorem

In a pure exchange economy with continuous, strictly convex, and locally nonsatiated preferences, $p \geq 0$ is a Walrasian equilibrium price vector if and only if $z(p) \leq 0$.

- Equilibrium allocations are simply Walrasian demands with the equilibrium price
- Note we also must have $z_l(p) = 0$ if $p_l > 0$ (this follows from Walras' law)
- If we adopt the stronger assumption that preferences are strongly monotone, then the equilibrium price must be strictly positive for each commodity, that is, $p \gg 0$ (otherwise consumers would demand infinite amounts of the free good)
- It then follows that with strongly monotonous preferences, equilibrium is simply characterized by the property $z(p) = 0$.

Existence of equilibrium in exchange economies

- Let us state a version of existence theorem valid for all positive price vectors:

Theorem

If $z(p)$ is a function defined for all $p \in \mathbb{R}^L_{+}$, is continuous, homogenous of degree zero, and satisfies Walras' law, then there is a price vector p^* such that $z(p^*) \leq 0$.

- This version is easy to prove (uses Brower's fixed point theorem, and will be discussed in the class), but the requirement of continuity is problematic: it will not hold at $p_l = 0$ with strongly monotonous preferences, because with such preferences demand of l goes to infinity at $p_l \to 0$
- There is a more applicable version of the existence therem: Proposition 17.C.1 in MWG
- Note that another assumption in the theorem is the requirement that demand is a function rather than correspondence. This is true with convex preferences (or with a large number of consumers - see Section 17.I in MWG)

Uniqueness of equilibrium (just brief remarks)

- Uniqueness is important in terms of the predictive power of the model
- Unfortunately, uniqueness in general equilibrium models can only be guaranteed under special circumstances
- One such case is: if all pairs of goods are gross substitutes of each other at all prices, then there is a unique equilibrium
- Without global uniqueness, we can still hope for local uniqueness (that is, each equilibrium price vector is unique in its neighbourhood)
- Local uniqueness holds generally in general equilibrium models (it is a generic property)
- More about uniqueness: MWG sections 17.D, 17.F

General equilibrium under uncertainty

- We represent uncertainty using states
- Assume that there are S distinct states, $s = 1, ..., S$
- As before, there are L physical commodities ($l = 1, ..., L$)
- We denote by $x = (x_{11}, ..., x_{L1}, ..., x_{1S}, ..., x_{LS}) \in \mathbb{R}^{LS}$ a *contingent commodity vector*
- Here x_{ls} represents the amount of physical commodity l obtained if state s is realized (a negative x_{ls} means oblication to deliver good l in state s)
- Endowments are also defined contingent on state: $\omega_i = (\omega_{11i}, ..., \omega_{L1i}, ..., \omega_{1Si}, ..., \omega_{LSi}) \in \mathbb{R}^{LS}$
- We represent preferences of i as a rational preference relation $\succsim_i$ defined on consumption set $X_i \subset \mathbb{R}^{LS}$
- Technologies are described by a state-contingent production possibilities set $Y_j \subset \mathbb{R}^{LS}$
- For simplicity, assume that ownership shares are state independent: $\theta_{ji} \geq 0$, $\Sigma_j \theta_{ji} = 1$

Discussion:

- Note that preferences are defined over contingent commodity vectors
- We can easily accomodate expected utility formulation. That is, let

$$x_i \succsim_i x_i' \iff \Sigma_s \pi_{si} u_{si}\left(x_{1si}, ..., x_{Lsi}\right) \geq \Sigma_s \pi_{si} u_{si}\left(x_{1si}', ..., x_{Lsi}'\right),$$

  where π_{si} is the probability (assigned by i) of state s, and u_{si} is i's Bernoully state-dependent utility function
- Convexity of preferences relates to risk aversion: if there is an expected utility representation, $\succsim_i$ is convex when Bernoully utility functions of i are concave
- Note that preferences are "ex-ante" preferences: they rank contingent commodities before uncertainty is resolved
- Decisions that are made before or after resolution of uncertainty can be distinguished by defining Y_j appropriately (similarly for consumption decisions by defining X_i)

Arrow-Debreu Model

- Arrow-Debreau model refers to a world, where all possible contingent commodities have well functioning markets
- These markets clear before the resolution of uncertainty
- Price p_{ls} is the price of an entitlement to one unit of commodity l in case state s realizes
- Seller of such an entitlement must deliver a unit of l if s realizes
- Existence of all these markets is of course a strong assumption
- The model is actually just a particular specification of the general equilibrium model that we have been analyzing

- Walrasian equilibrium in this context is called an Arrow-Debreu equilibrium:

Definition

An allocation $(x_1^*, ..., x_I^*, y_1^*, ..., y_J^*) \in X_1 \times \cdot ... \cdot \times X_I \times Y_1 \times \cdot ... \cdot \times Y_J \subset \mathbb{R}^{LS(I+J)}$ and a price vector $p = (p_{11}, ..., p_{LS}) \in \mathbb{R}^{LS}$ constitute an Arrow-Debreu equilibrium if:

1. For every j, y_j^* maximizes profits in Y_j; that is

$$p \cdot y_j \leq p \cdot y_j^* \text{ for all } y_j \in Y_j.$$

2. For every i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\left\{ x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^* \right\}.$$

3.

$$\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$$

- Note that this is in no way different to our earlier definition of Walrasian equilibrium
- In particular, welfare theorems and existence/uniqueness issues discussed ealier apply to this context
- Since all commodities are traded ex-ante contingent on all realizations of s, there is no scope for ex-post trade. Ex-ante Pareto-optimality implies ex-post Pareto-optimality.
- But clearly the existence of markets for all contingent commodities is very strong assumption indeed
- Fortunately, we can do with less
- In fact, we'll see that contingent markets for one physical good for all states is enough
- Intuition: one commodity is enough to transfer wealth across states to reach the same outcome (with appropriate ex-post trade after the realization of state)

Sequential Trade

- So, let us consider sequential trade, where there is contingent trade on commodity 1 at date $t = 0$, and ex-post trade on all commodities at date $t = 1$ (after uncertainty has resolved)
- For simplicity, assume pure exchange
- Denote the prices of the contingent commodites traded at $t = 0$ by $q = (q_1, ..., q_S) \in \mathbb{R}^S$
- Denote the prices of all commodities at $t = 1$ (spot prices) by $p = (p_1, ..., p_S) \in \mathbb{R}^{LS}$
- Denote by $(z_{1i}, ..., z_{Si}) \in \mathbb{R}^S$ the trades of i of the contingent commodity
- Denote by $(x_{1i}, ..., x_{Si}) \in \mathbb{R}^{LS}$ the spot market consumption plan of i
- Assume that consumers have correct expectations of spot prices when they trade on forward markets

- Denoting by $U_i(\cdot)$ the utility for $\succsim_i$, consumer i's problem is:

$$\underset{(z_{1i},...,z_{Si}),(x_{1i},...,x_{Si})}{Max} U_i\left(x_{1i}, ..., x_{Si}\right)$$

$$\begin{aligned} s.t. \text{ (i) } \Sigma_s q_s z_{si} &\leq 0 \\ \text{(ii) } p_s \cdot x_{si} &\leq p_s \cdot \omega_{si} + p_{1s} z_{si} \text{ for every } s \end{aligned}$$

- Assuming consumers' expectations of spot prices are rational, the equilibrium can be stated as:

Definition

A collection formed by $q = (q_1, ..., q_S) \in \mathbb{R}^S$, $p = (p_1, ..., p_S) \in \mathbb{R}^{LS}$, $z_i^* = (z_{1i}^*, ..., z_{Si}^*) \in \mathbb{R}^S$ for all $i = 1, ..., I$, and $x_i^* = (x_{1i}^*, ..., x_{Si}^*) \in \mathbb{R}^{LS}$ for all $i = 1, ..., I$, constitutes a Radner equilibrium if:

1. For every i, consumption plans z_i^*, x_i^* solve the problem of the previous slide
2. $\Sigma_i z_{Si}^* \leq 0$ and $\Sigma_i x_{si}^* \leq \Sigma_i \omega_{si}$ for every s

- Note that trade takes place sequentially, but the consumers take into account period $t = 1$ trades in optimizing period $t = 0$ trades
- It is natural to normalize $p_{1s} = 1$ for each s, so a unit of the $s-$contingent commodity pays off 1 dollar in state s

- Importantly, in the equilibrium defined above, allocations are identical to the Arrow-Debreu equilibrium:

Theorem

(i) If $x^* \in \mathbb{R}^{LSI}$ *and* $p = (p_1, ..., p_S) \in \mathbb{R}^{LS}_{++}$ *constitute an Arrow-Debreau equilibrium, then there are* $q \in \mathbb{R}^{S}_{++}$ *and* $z^* \in \mathbb{R}^{SI}$ *such that* (q, p, z^*, x^*) *constitute a Radner equilibrium*

(ii) If $(q, p, z^*, x^*) \in \mathbb{R}^{S}_{++} \times \mathbb{R}^{LS}_{++} \times \mathbb{R}^{SI} \times \mathbb{R}^{LSI}$ *constitute a Radner equilibrium, then there are multipliers* $(\mu_1, ..., \mu_S) \in \mathbb{R}^{S}_{++}$ *such that* x^* *and* $(\mu_1 p_1, ..., \mu_S p_S) \in \mathbb{R}^{LS}_{++}$ *constitute an Arrow-Debreu equilibrium.*

- Idea of proof: budget sets are identical in the two settings
- Multiplier μ_s is to be interpreted as the value, at $t = 0$, of a dollar at $t = 1$ in state s.

Asset Markets

- The role of contingent commodities above is to transfer wealth ex-ante across future states of world
- In reality, assets do not look quite like the ones defined above
- So, the question we ask here is, can we do the same with other kind of assets?
- The answer will be yes, if the asset markets are complete (in a sense to be defined)
- We continue to assume: two dates $t = 0$ and $t = 1$, where consumption takes place only at date $t = 1$
- This could be generalized with no conceptual difficulties

Definition

A unit of asset, or security, is a title to receive an amount r_s of good 1 at date $t = 1$ if state s occurs. An asset is characterized by its return vector $r = (r_1, ..., r_S) \in \mathbb{R}^S$.

- Examples of assets:
  - $r = (1, ..., 1)$ is a commodity future
  - $r = (0, ..., 0, 1, 0, ..., 0)$ is contingent commodity discussed earlier (an Arrow security)
  - $r = (\max\{0, r_1 - c\}, ..., \max\{0, r_S - c\})$ is an option, where $r = (r_1, ..., r_S)$ is a primary asset and c is the strike price (options are derivative assets, because they derive their value from that of another asset)
- We can summarize the asset structure in a $S \times K$ return matrix R:

$$R = \begin{bmatrix} r_{11}, ..., r_{1K} \\ .. \\ r_{S1}, ..., r_{SK} \end{bmatrix}$$

- Return of portfolio z is thus Rz

- Given K assets traded at prices $q = (q_1, ..., q_K)$ at $t = 0$, the equilibrium is:

Definition

A collection formed by $q = (q_1, ..., q_K) \in \mathbb{R}^K$, $p = (p_1, ..., p_S) \in \mathbb{R}^{LS}$, $z_i^* = (z_{1i}^*, ..., z_{Ki}^*) \in \mathbb{R}^K$ for all $i = 1, ..., I$, and $x_i^* = (x_{1i}^*, ..., x_{Si}^*) \in \mathbb{R}^{LS}$ for all $i = 1, ..., I$, constitutes a Radner equilibrium if:

1. For every i, consumption plans z_i^*, x_i^* solve

$$\underset{(z_{1i},...,z_{Ki}),(x_{1i},...,x_{Si})}{Max} U_i(x_{1i}, ..., x_{Si})$$

$$\begin{aligned} s.t.\ \text{(i)}\ \Sigma_k q_k z_{ki} &\leq 0 \\ \text{(ii)}\ p_s \cdot x_{si} &\leq p_s \cdot \omega_{si} + \Sigma_k p_{1s} z_{ki} r_{sk} \text{ for every } s \end{aligned}$$

2. $\Sigma_i z_{ki}^* \leq 0$ and $\Sigma_i x_{si}^* \leq \Sigma_i \omega_{si}$ for every k and s

- Return matrix restricts the set of possible asset price vectors. This result is an expression of a very important principle: in equilibrium, asset prices must be arbitrage-free.

Theorem

Assume that every return vector is nonnegative and nonzero. Then, for every price vector $q \in \mathbb{R}^K$ *arising in a Radner equilibrium, we can find multipliers* $\mu = (\mu_1, ..., \mu_S) \geq 0$*, such that* $q_k = \Sigma_s \mu_s r_{sk}$ *for all* k*. In matrix notation,* $q^T = \mu \cdot R$*.*

- Two versions of proof given in MWG, section 19.E

- Let us now give the key definition:

Definition

An asset structure with an $S \times K$ return matrix R is complete if $rank\,(R) = S$, that is, if there is a subset of S assets with linearly independent returns.

- With S linearly independent returns, one can obtain any return profile by choosing portfolio appropriately
- Thus, with a complete asset structure, agents are again unrestricted in their wealth transfers across states
- It is therefore not surprising, that we again end up in the same allocation as in Arrow-Debreau equilibrium:

Theorem

Suppose that the asset structure is complete. Then:
(i) If $x^* \in \mathbb{R}^{LSI}$ *and* $p = (p_1, ..., p_S) \in \mathbb{R}^{LS}_{++}$ *constitute an Arrow-Debreau equilibrium, then there are asset prices* $q \in \mathbb{R}^{K}_{++}$ *and portfolio plans* $z^* \in \mathbb{R}^{KI}$ *such that* (q, p, z^*, x^*) *constitute a Radner equilibrium*
(ii) If $(q, p, z^*, x^*) \in \mathbb{R}^{K}_{++} \times \mathbb{R}^{LS}_{++} \times \mathbb{R}^{KI} \times \mathbb{R}^{LSI}$ *constitute a Radner equilibrium, then there are multipliers* $(\mu_1, ..., \mu_S) \in \mathbb{R}^{S}_{++}$ *such that* x^* *and* $(\mu_1 p_1, ..., \mu_S p_S) \in \mathbb{R}^{LS}_{++}$ *constitute an Arrow-Debreu equilibrium.*

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- Idea of proof as before (see MWG)
- Again, multiplier μ_s is the value, at $t = 0$, of a dollar at $t = 1$ in state s

- An important principle is that of *pricing by arbitrage*.
- To see how it works, assume there is a complete asset structure
- Then there must be some S assets with linearly independent returns
- Use their prices q_k to deduce the state multipliers $\mu = (\mu_1, ..., \mu_S)$ (solve μ from S linear equations $q_k = \Sigma_s \mu_s r_{sk}$, $k = 1, ..., S$)
- Once you have the state multipliers, price of any other asset k' with return vector $r_{k'}$ must be $q_{k'} = \Sigma_s \mu_s r_{sk'}$
- Otherwise there would be an arbitrage opportunity. Can you see why? It is a good excercise to work this out in more detail
- Note that state multipliers do not depend directly on state probabilities
- This issue is at the heart of a large body of finance theory
- With incomplete markets, things are much more complicated