Microeconomic Theory I

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Lecture 6: Topics on decision making under uncertainty

- In this lecture, we'll continue with the theory of choice under uncertainty
- In the previous lecture, we considered the comparison of risk attitudes (e.g. measures of absolute and relative risk aversion)
- Now, we will compare payoff distributions with a fixed preference relation
- The central concepts are first and second order stochastic dominance relations
- Then we will consider some applications of Expected Utility Theory:
  - Portfolio choice
  - Consumption and saving
- Most of this lecture is covered in chapters 1-4 of Gollier (2001).
- Much of this can also be found in MWG Chapter 6

Comparison of payoff distributions in terms of return and risk

- Here we aim at comparison of risks for a fixed utility function
- This leads to notions of first and second order stochastic dominance
- These relate to comparison of return and risk of a payoff distribution, respectively (a viewpoint emphasized in MWG)
- Here we emphasize another viewpoint: A stochastic dominance (of some order) means that a distribution dominates another one in terms of expected utility for a certain class of utility functions

- If Ω is the set of utility functions that are of interest, we are after characterizations for distributions $dF_1(x)$ and $dF_2(x)$ to guarantee that

$$\int u(x)\,dF_1(x) \geq \int u(x)\,dF_2(x) \text{ for all } u(x) \in \Omega. \qquad (1)$$

- Since the set Ω may be a large set, it would be helpful if we could obtain the desired conclusion based on evaluating the integrals in (1) for a smaller set of functions.
- The key: linearity of integrals. If

$$\int u(x)\,dF_1(x) \geq \int u(x)\,dF_2(x) \text{ and}$$

$$\int \widehat{u}(x)\,dF_1(x) \geq \int \widehat{u}(x)\,dF_2(x), \text{ then}$$

$$\int [\alpha u(x) + \beta \widehat{u}(x)]\,dF_1(x) \geq \int [\alpha u(x) + \beta \widehat{u}(x)]\,dF_2(x).$$

- We want to find a basis for Ω, i.e. a set of functions $\{b(x,\theta)\}_{\theta\in\Theta}$, such that each $u \in \Omega$ is a linear combinations of those functions.
- If Θ is uncountable, then linear combinations of functions in $\{b(x,\theta)\}_{\theta\in\Theta}$, $\Theta \subset \mathbb{R}$, are given by integrals over Θ.
- Hence we want to find a set of functions $\{b(x,\theta)\}_{\theta\in\Theta}$ such that for all $u \in \Omega^1$, we have:

$$u(x) = \int b(x,\theta)\,dH(\theta)$$

for some "distribution" $dH(\theta)$:

$$H : \Theta \to \mathbb{R}^+,$$

$$\int_\Theta dH(\theta) = 1.$$

- If we know that

$$\int b(x,\theta)\,dF_1(x) \geq \int b(x,\theta)\,dF_2(x) \text{ for all } \theta,$$

- then we know immediately that

$$\int u(x)\,dF_1(x) = \int \int b(x,\theta)\,dH(\theta)\,dF_1(x)$$

$$= \int \int b(x,\theta)\,dF_1(x)\,dH(\theta) \geq \int \int b(x,\theta)\,dF_2(x)\,dH(\theta)$$

$$= \int \int b(x,\theta)\,dH(\theta)\,dF_2(x) = \int u(x)\,dF_2(x).$$

- From now on, we will specialize this approach to two classes of Bernoulli utility functions.

First order stochastic dominance

- For this notion, the utility functions of interest are all increasing functions, i.e. we consider

$$\Omega^1 = \left\{u(x) : u'(x) \geq 0\right\}.$$

- We will assume that $x \in [\underline{x}, \overline{x}]$ and that u is differentiable throughout.

Definition

A distribution $dF_1(x)$ first order stochastically dominates $dF_2(x)$ if

$$\int u(x)\, dF_1(x) \geq \int u(x)\, dF_2(x) \text{ for all } u(x) \in \Omega^1.$$

- For this class of functions, use indicator functions as the basis:

$$b(x, \theta) = I(x \geq \theta),$$

where $I(x \geq \theta) = \begin{cases} 1 \text{ if } x \geq \theta, \\ 0 \text{ if } x \leq \theta. \end{cases}$

- To see this, recall from the fundamental theorem of differential and integral calculus that

$$u(x) = u(\underline{x}) + \int_{\underline{x}}^{x} u'(\theta)\, d\theta.$$

- There is no loss of generality in normalizing all the utility functions to have $u(\underline{x}) = 0$ (why?). Hence

$$u(x) = \int_{\underline{x}}^{x} u'(\theta)\, d\theta = \int_{\underline{x}}^{\overline{x}} I(x \geq \theta)\, u'(\theta)\, d\theta.$$

- Hence the weight function $dH(\theta)$ is given by $u'(\theta)\, d\theta$.
- With this basis, we can easily characterize when $dF_1(x)$ first order stochastically dominates $dF_2(x)$.
- We must have

$$\int_{\underline{x}}^{\overline{x}} I(x \geq \theta)\, dF_1(x) \geq \int_{\underline{x}}^{\overline{x}} I(x \geq \theta)\, dF_2(x) \text{ for all } \theta.$$

- In other words,

$$\int_{\theta}^{\overline{x}} dF_1(x) \geq \int_{\theta}^{\overline{x}} dF_2(x) \text{ for all } \theta, \text{ or}$$

$$1 - F_1(\theta) \geq 1 - F_2(\theta) \text{ for all } \theta, \text{ or}$$

$$F_1(\theta) \leq F_2(\theta) \text{ for all } \theta.$$

- Thus we have proved the following proposition.

Proposition

$dF_1(x)$ first order stochastically dominates $dF_2(x)$ if and only if

$$F_1(x) \leq F_2(x) \text{ for all } x \in [\underline{x}, \overline{x}].$$

- Another characterization:

Proposition

- *Random variable $\widetilde{y}$ first order stochastically dominates random variable $\widetilde{x}$ if and only if we can write*

$$\widetilde{y} = \widetilde{x} + \widetilde{z},$$

where $\widetilde{z}$ is a random variable with $Pr[\widetilde{z} \leq 0 \,|\, \widetilde{x}] = 0$.

Second order stochastic dominance

- For this notion, the relevant class of Bernoulli utility functions is given by $\Omega^2 = \{u(x) : u'(x) \geq 0, u''(x) \leq 0\}$.

Definition

A distribution $dF_1(x)$ second order stochastically dominates $dF_2(x)$ if

$$\int u(x)\, dF_1(x) \geq \int u(x)\, dF_2(x) \text{ for all } u(x) \in \Omega^2.$$

- $b(x, \theta) = \min(x, \theta)$ form a basis for Ω^2.
- To see this, let

$$u(x) = \int_{\underline{x}}^{\overline{x}} \min(x, \theta)\, dH(\theta). \quad (2)$$

- Integration by parts yields:

$$\int_{\underline{x}}^{\overline{x}} \min(x,\theta)\, dH(\theta) = |_{\underline{x}}^{\overline{x}} \min(x,\theta)\, H(\theta) - \int_{\underline{x}}^{\overline{x}} I(x \geq \theta)\, H(\theta)\, d\theta$$

$$= xH(\overline{x}) - \underline{x}H(\underline{x}) - \int_{\underline{x}}^{x} H(\theta)\, d\theta.$$

- From this:

$$u'(x) = H(\overline{x}) - H(x),$$
$$u''(x) = -H'(x),$$

- so this works when the weight function is

$$dH(\theta) = -u''(\theta)\, d\theta.$$

- The remaining step is to derive the implications from

$$\int_{\underline{x}}^{\overline{x}} \min(x,\theta)\, dF_1(x) \geq \int_{\underline{x}}^{\overline{x}} \min(x,\theta)\, dF_2(x) \text{ for all } \theta. \qquad (3)$$

- Integrating the left hand side by parts yields:

$$|_{\underline{x}}^{\overline{x}} \min(x,\theta)\, F_1(x) - \int_{\underline{x}}^{\overline{x}} I(x \leq \theta)\, F_1(x)\, dx$$

$$= \theta - \int_{\underline{x}}^{\theta} F_1(x)\, dx.$$

- Similarly for the right hand side, we get:

$$\theta - \int_{\underline{x}}^{\theta} F_2(x)\, dx.$$

- Hence we get that 3 hold if and only if

$$\theta - \int_{\underline{x}}^{\theta} F_1(x)\, dx \geq \theta - \int_{\underline{x}}^{\theta} F_2(x)\, dx \text{ for all } \theta \text{ or}$$

$$\int_{\underline{x}}^{\theta} F_1(x)\, dx \leq \int_{\underline{x}}^{\theta} F_2(x)\, dx \text{ for all } \theta.$$

- We have thus proved the following proposition.

Proposition

$dF_1(x)$ second order stochastically dominates $dF_2(x)$ if and only if

$$\int_{\underline{x}}^{\theta} F_1(x)\, dx \leq \int_{\underline{x}}^{\theta} F_2(x)\, dx \text{ for all } \theta.$$

- Another characterization:

Proposition

- *Random variable $\widetilde{x}$ second order stochastically dominates random variable $\widetilde{y}$ if and only if we can write*

$$\widetilde{y} = \widetilde{x} + \widetilde{z},$$

where $E[\widetilde{z}\,|\widetilde{x}] \leq 0$.

Standard Portfolio Choice

- Consider risk averse decision maker.
- Initial wealth w_0.
- Decision problem: How much to invest in safe versus risky assets?
- No short sales allowed.
- $(1+r)$ riskless return
- $(1+\widetilde{x})$ the random return on the risky investment.
- Denote the amount of risky investment by $0 \leq \alpha \leq w_0$, and thus the safe investment is $(w_0 - \alpha)$.
- Final wealth of the decision maker:

$$(w_0 - \alpha)(1+r) + \alpha(1+\widetilde{x}) = w_0(1+r) + \alpha(\widetilde{x} - r).$$

- Assume strictly concave, strictly increasing twice differentiable Bernoulli utility function $u(w)$.
- Expected utility from a risky investment α:

$$v(\alpha) = Eu(w_0(1+r) + \alpha(\widetilde{x} - r)).$$

- $v(\alpha)$ is a strictly concave function of α if $\Pr(\widetilde{x} = r) < 1$ since

$$v''(\alpha) = E\left((\widetilde{x} - r)^2 u''(w_0(1+r) + \alpha(\widetilde{x} - r))\right) < 0.$$

- FOC is thus also sufficient for maximum.
- The first order condition for interior solutions (i.e. for solutions where $0 < \alpha < w_0$):

$$v'(\alpha) = E\left[(\widetilde{x} - r)\, u'(w_0(1+r) + \alpha(\widetilde{x} - r))\right] = 0.$$

- For $\alpha = 0$, it must be that

$$v'(0) = E\left[(\tilde{x} - r)\, u'(w_0(1+r))\right] \le 0.$$

- Since $u'(w_0(1+r))$ is independent of $\tilde{x}$, the above condition is equivalent to

$$u'(w_0(1+r))\, E(\tilde{x} - r) \le 0.$$

- Hence a necessary condition for no risky investments is that the expected value of the investment be no larger than the safe return.
- This is also a sufficient condition (exercise: why?).
- Thus all decision makers, risk averse or not, invest some positive amount in risky assets if their expected return is larger than the safe rate.

- Consider next two risk averse decision makers, u_1 and u_2.
- Suppose that u_1 is more risk averse than u_2.
- Then $u_1(x) = \phi(u_2(x))$ for some concave function ϕ.
- We want to see how the optimal portfolio choices of u_1 and u_2 can be compared.
- Denote the optimal risky investments by α_1 and α_2 respectively. From the first order condition for u_2, we have:

$$v_2'(\alpha_2) = E\left[(\tilde{x} - r)\, u_2'(w_0(1+r) + \alpha_2(\tilde{x} - r))\right] = 0. \quad (4)$$

- To see how the optimal risky investment of u_1 relates to α_2, we evaluate the derivative of $v_1(\cdot)$ at $\alpha = \alpha_2$.

$$v_1'(\alpha_2) = \frac{d}{d\alpha} E\phi(u_2(w_0(1+r) + \alpha_2(\tilde{x} - r)))$$

$$= E(\tilde{x} - r)\, \phi'(u_2(w_0(1+r) + \alpha_2(\tilde{x} - r)))\, u_2'(w_0(1+r) + \alpha_2(\tilde{x} - r$$

- Since $\phi'' \le 0$, we know that

$$(\tilde{x} - r)\, \phi'(u_2(w_0(1+r) + \alpha_2(\tilde{x} - r))) \le (\tilde{x} - r)\, \phi'(u_2(w_0(1+r)))$$

- for all $\tilde{x}$.
- To see this note that for $\tilde{x} < r$,

$$\phi'(u_2(w_0(1+r) + \alpha_2(\tilde{x} - r))) \ge \phi'(u_2(w_0(1+r)))$$

- by the concavity of ϕ and similarly for $\tilde{x} > r$,

$$\phi'(u_2(w_0(1+r) + \alpha_2(\tilde{x} - r))) \le \phi'(u_2(w_0(1+r)))$$

- and hence the claim follows.

- But then we know that

$$v_1'(\alpha_2) \le E(\tilde{x} - r)\, \phi'(u_2(w_0(1+r)))\, u_2'(w_0(1+r) + \alpha_2(\tilde{x} - r))$$

$$= \phi'(u_2(w_0(1+r)))\, E(\tilde{x} - r)\, u_2'(w_0(1+r) + \alpha_2(\tilde{x} - r)) = 0,$$

- where the last equality follows from (4). Thus by the concavity of $v_1(\alpha)$, we know that $\alpha_1 \le \alpha_2$.

Proposition

If u_1 is more risk averse than u_2, then $\alpha_1 \le \alpha_2$ in the standard portfolio problem.

- This proposition also yields an immediate corollary for risky investment as a function of initial wealth.

Proposition

If u satisfies decreasing absolute risk aversion, then $\alpha(w_0) \le \alpha(w_0')$ whenever $w_0 < w_0'$.

Consumption and Savings

- Start with the simplest deterministic two-period model, and derive conclusions for optimal savings and consumption.
- Additively separable utility function.
- In other words, the consumer has a separate Bernoulli utility function for the consumption in each period $t = 0, 1$.
- The consumer receives wealth w_0 and w_1 respectively in the two periods.
- She can borrow and lend as she wishes at the risk free rate r.

- If we let s denote the savings by the consumer, then the optimization problem can be written as

$$\max_s\ u_0(w_0 - s) + u_1(w_1 + s(1+r)).$$

- Observe that we can allow for negative saving (i.e. borrowing) in this model, but we require that consumption be positive in both periods (i.e. $-\frac{w_1}{1+r} \le s \le w_0$).
- Assume throughout that $u_i(\cdot)$ are strictly concave and twice continuously differentiable for $i = 0, 1$.
- Hence if we let

$$v(s) = u_0(w_0 - s) + u_1(w_1 + s(1+r)),$$

- we see immediately that $v''(s) < 0$.

- This allows us again to locate optimal savings levels from the first order conditions.
- The optimal level of savings s^* is characterized by

$$v'(s^*) = -u_0'(w_0 - s^*) + (1+r)\,u_1'(w_1 + s^*(1+r)) = 0.$$

- If $u_0 = u_1 = u$ and $r = 0$, we see the most clearly how savings are used to smooth consumption across periods.
- From

$$u'(w_0 - s^*) = u'(w_1 + s^*),$$

- we conclude by the strict concavity of u that

$$w_0 - s^* = w_1 + s^*.$$

- Hence the consumption levels in the two periods are identical.
- The other main motive of saving is to increase wealth.
- This effect can obviously only be seen when $r > 0$.
- Again in the case where $u_0 = u_1 = u$, we get

$$u'(w_0 - s^*) = (1+r)\,u'(w_1 + s^*(1+r)).$$

- By concavity of u, we see that consumption in the second period is larger (since the marginal utility is lower) than in the first period.
- Hence the consumer is willing to sacrifice some of the consumption smoothing for increases in wealth.

- Finally, we can totally differentiate the first order condition with respect to s and w_i to get

$$\frac{ds^*}{dw_0} = \frac{u_0''(w_0 - s^*)}{\left[u_0''(w_0 - s^*) + (1+r)^2 u_1''(w_1 + s^*(1+r))\right]} > 0,$$

$$\frac{ds^*}{dw_1} = \frac{-u_1''(w_1 + s^*(1+r))}{\left[u_0''(w_0 - s^*) + (1+r)^2 u_1''(w_1 + s^*(1+r))\right]} < 0.$$

- Hence an increase in the first period income increases savings, and an increase in the second period income decreases savings.
- With these preliminaries in place, we can analyse the optimal savings problem in a world of uncertainty.

- The question that we ask is whether the optimal savings are larger in a model where the second period income is random than in the deterministic model.

Definition

An agent is prudent if adding an uninsurable zero mean risk to the second period income increases the savings.

- To characterize prudent utility functions, let $\widetilde{w}_1 = w_1 + \widetilde{x}$, where $\widetilde{x}$ is assumed to be uninsurable and $E\widetilde{x} = 0$.
- Denote the new expected utility from savings s by:

$$V(s) = u_0(w_0 - s) + Eu_1(w_1 + s(1+r) + \widetilde{x}).$$

- $V(s)$ inherits the curvature of the u_i functions.

- Analyze comparative static questions by evaluating the derivative of $V(s)$ at point s^* such that $v'(s^*) = 0$, i.e. at the optimal savings level of the deterministic model.
- Observe that $V'(s^*) \geq 0$ if

$$Eu_1'(w_1 + s^*(1+r) + \widetilde{x}) \geq u_1'(w_1 + s^*(1+r)). \qquad (5)$$

- Notice that on the left hand side of the inequality, we have the expected utility from a random variable.
- On the right hand side, we have the utility from the expected value of the random variable.
- This is exactly the definition of a risk loving utility function since w_1 and $\widetilde{x}$ are arbitrary.
- As risk loving functions are convex, we deduce that (5) holds for all w_1 and $\widetilde{x}$ if and only if u_1' is convex.

- Hence we have proved the following proposition.

Proposition

An agent is prudent if and only if u_1' is convex.

- From this point on, we could develop a theory for comparing prudence of different individuals or the prudence of a given individual at various wealth levels.
- Much of this theory has been done by Miles Kimball, and the central concept for the analysis is the coefficient of absolute prudence:

$$P(w) = \frac{-u'''(w)}{u''(w)}.$$

- We conclude this section on noting a relationship between prudence and the coefficient of absolute risk aversion. It is easy to show that:

$$\frac{d}{dw} r^A(w) = r^A(w) \left[r^A(w) - P(w) \right].$$

- Hence there are two arguments for believing in the prevalence of prudent utility functions.
- First of all, there is direct econometric evidence on the savings behavior of individuals with various degrees of uninsurable risk positions.
- Second, there is overwhelming empirical support for decreasing absolute risk aversion.
- As the formula above indicates, DARA is only possible for prudent utility functions.