Microeconomic Theory I

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Finnish Doctoral Programme in Economics

Autumn 2009

Lecture 5: Choice under Uncertainty

- Decisions are typically made uncer uncertainty
- What is uncertainty? How does it affect decisions? How can it be quantified?
- How should we formalize choice under uncertainty?
- In this and the next lecture we will discuss various issues related to individual decision making under uncertainty
- The most important development in this lecture is the expected utility theorem
- Material for this lecture: MWG chapter 6. Kreps (1988) goes deeper in the theoretical development, and Gollier (2001) provides useful additional material on the more applied side

Approaches to Probability

- How would you assess the following probabilities?
  - ‘The coin toss results in Heads’
  - ‘Social Democrats will be the largest party in the next election’
  - ‘Rome is more northern than Madrid’
- Classical view: Probability of an event is the long run frequency of the occurrence of the event in a sequence of independent experiments
- Subjectivist view: There is no other meaning to the probability of an event except as a feature of a decision maker’s preferences in a choice situation.
- In the subjectivist view, probability can be deduced from choice behavior.
- In classical view, only the first statement above can have a probabilistic meaning whereas in the subjectivist view all of them can have a probabilistic interpretation.

Probability

- Consider first a finite set of possible outcomes or consequences C.
- To talk about random experiments on C, we define events.
- Events form a family of subsets of C, $\mathcal{A}$.
- $\mathcal{A}$ is assumed to satisfy:

$$i)\ C \in \mathcal{A}.$$

$$ii)\ A \in \mathcal{A} \text{ implies that } A^C \in \mathcal{A}.$$

$$iii)\ A_i \in \mathcal{A} \text{ for } i = 1, 2, ... \text{ implies that } \cup_{i=1}^{\infty} A_i \in \mathcal{A}.$$

- Probability is a non-negative real valued function on $\mathcal{A}$.
- In general, we require:

$$\begin{aligned} i)\ P(\emptyset) &= 0, \\ ii)\ P(C) &= 1, \\ iii)\ P(A \cup B) &= P(A) + P(B) \text{ if } A \cap B = \emptyset, \\ iv) P\left(\cup_{i=1}^{\infty} A_i\right) &= \Sigma_{i=1}^{\infty} P(A_i) \\ \text{if } A_i \cap A_j &= \emptyset \text{ for all } i \neq j. \end{aligned}$$

- With a finite $C = \{c_1, ..., c_N\}$, it is natural to take $\mathcal{A} = 2^C$.
- Define:

$$\mathcal{L} = \{(p_1, ..., p_N) : \Sigma_i p_i = 1 \text{ and } p_i \geq 0 \text{ for all } i\}$$

- We call $\mathcal{L}$ the set of *simple lotteries*.

Objective in this class: Find useful representations for preferences defined on $\mathcal{L}$

- Distinction made in literature:
  - Probabilities are exogenously given $\leftrightarrow$ risk.
  - Probabilities subjectively evaluated $\leftrightarrow$ uncertainty.
- Risk: von Neumann-Morgenstern.
- Uncertainty: Savage.
- Combination of the two (Horse-Race-Roulette-Wheel): Anscombe-Aumann.
- We mostly focus on vN-M formulation

von Neumann - Morgenstern approach:

- We assume here for simplicity that C is finite.
- We build on three axioms:
- **Axiom 1:** $\succsim$ is a rational preference relation on $\mathcal{L}$.
- Observe that $\mathcal{L}$ is a convex set (can you prove this?) and hence it makes sense to talk about *compound lotteries*.
- Take $L, L' \in \mathcal{L}$. A compound lottery $L^{\alpha} \in \mathcal{L}$ is obtained by setting:

$$L^{\alpha} = \alpha L + (1 - \alpha) L'.$$

- The meaning of this is that if $(p_1, ..., p_N)$ gives the probabilities of c_n under L, and $(p'_1, ..., \acute{} p'_N)$ gives the probabilities under L', then the probabilities under L^{α} are given by $(\alpha p_1 + (1 - \alpha p'_1), ..., \alpha p_N + (1 - \alpha) p_N)$.

- We are implicitly assuming here that compound lotteries are equivalent to induced simple lotteries.
- Denote by δ_{c_n} the degenerate lottery that yields c_n with probability 1.

Continuity of preferences

- We formulate the continuity axiom slightly differently from abstract choice theory:
- **Axiom 2:** (Archimedean Axiom) Take $L, L', L'' \in \mathcal{L}$ such that $L \succ L' \succ L''$. Then the sets $\{\alpha \in [0,1] : \alpha L + (1-\alpha) L'' \succeq L'\}$ and $\{\alpha \in [0,1] : \alpha L + (1-\alpha) L'' \preceq L'\}$ are closed.
- Together with completeness, this implies that small changes in probabilities do not affect the orderings of lotteries.
- Notice that this implies a bound on how good or bad some outcomes can be. In particular, no outcome can be infinitely painful.

- Continuity and rationality imply the existence of a continuous representation $U : \mathcal{L} \to \mathbb{R}$.
- $\mathcal{L}$ is in general too complicated for analysis
- We want to deduce the existence of a utility function $u : C \to \mathbb{R}$ that represents $\succsim$ in a straightforward manner: $U(L) = E_L u$.
- We call U the von Neumann-Morgenstern utility function and u the Bernoulli utility function of the decision maker.
- The next axiom is the key to the representation result.

Independence axiom

- **Axiom 3:** (Independence Axiom) For any $L, L' \in \mathcal{L}$, we have

$$L \succeq L' \Leftrightarrow \alpha L + (1-\alpha) L'' \succeq \alpha L' + (1-\alpha) L''$$

  for all $\alpha \in [0,1]$ and for all $L'' \in \mathcal{L}$.
- It should be clear that an axiom of this type makes no sense for choice under certainty.
- Does it make sense for theory of choice under uncertainty?
- Normative vs. positive implications.

Expected Utility Theorem

Theorem

(Expected Utility Theorem) A rational preference $\succeq$ *on* $\mathcal{L}$ *satisfies the Archimedean and Independence axiom if and only if there exists a utility function* $u : C \to \mathbb{R}$ *such that*

$$L \succeq L' \Leftrightarrow \Sigma_{n=1}^{N} p_n u_n \geq \Sigma_{n=1}^{N} p'_n u_n.$$

Furthermore, if u *and* u' *are such representations, then* $u' = \beta u + \gamma$ *where* $\beta > 0$.

Proof of Expected Utility Theorem

i) It is easy to verify that the axioms must be satisfied if a representation exists.
ii) Let us show that the axioms imply the existence of such a representation. Denote by δ_c the degenerate lottery that assigns probability 1 on consequence $c \in C$.
Consider δ_{c_n}. Since C is finite, there exist c_w and c_b such that $\delta_{c_w} \preceq \delta_{c_n}$ for all n and $\delta_{c_n} \preceq \delta_{c_b}$ for all n. By independence axiom, $\delta_{c_w} \preceq L$ for all $L \in \mathcal{L}$ and $\delta_{c_b} \succeq L$ for all $L \in \mathcal{L}$. If $\delta_{c_w} \sim \delta_{c_b}$, we can take u to be any constant function and the theorem is proved. Assume thus that $\delta_{c_w} \prec \delta_{c_b}$.
Let $u\left(\delta_{c_w}\right) = 0$ and $u\left(\delta_{c_b}\right) = 1$.

Proof (cont.)

We argue next that for each $L \in \mathcal{L}$, there is a unique $\alpha(L)$ such that $L \sim \alpha(L)\,\delta_{c_b} + (1-\alpha(L))\,\delta_{c_w}$. First note by the independence axiom and the assumption that $\delta_{c_w} \prec \delta_{c_b}$ that

$$p\delta_{c_b} + (1-p)\,\delta_{c_w} \succ q\delta_{c_b} + (1-q)\,\delta_{c_w} \text{ if } p > q.$$

By Archimedean axiom, $\{\alpha : \alpha\delta_{c_b} + (1-\alpha)\,\delta_{c_w} \succeq L\}$ and $\{\alpha : \alpha\delta_{c_b} + (1-\alpha)\,\delta_{c_w} \preceq L\}$ are closed.
By the completeness of $\succeq$, the union of these sets in [0,1] and since $[0,1]$ is a connected, the intersection of the sets must be nonempty. By the observation above, the intersection must be a singleton and we can write $\alpha(L)$ for the unique element in the intersection.

Proof (cont.)

To show that $\alpha(L)$ takes the form $\Sigma_{n=1}^{N} p_n u_n$ for some $u : C \to \mathbb{R}$, all we need to show is that for all L, L' and $\gamma \in [0,1]$ we have:

$$\alpha\left(\gamma L + (1-\gamma) L'\right) = \gamma\alpha(L) + (1-\gamma)\alpha\left(L'\right).$$

(Why is this sufficient?).
This is the case since

$$\begin{aligned} & \gamma L + (1-\gamma) L' \\ \sim\ & \gamma[\alpha(L)\delta_{c_b} + (1-\alpha(L))\delta_{c_w}] + (1-\gamma)\left[\alpha\left(L'\right)\delta_{c_b} + \left(1-\alpha\left(L'\right)\right)\delta_{c_w}\right] \\ \sim\ & [\gamma\alpha(L) + (1-\gamma)\alpha\left(L'\right)]\delta_{c_b} + [\gamma(1-\alpha(L) + (1-\gamma)\left(1-\alpha\left(L'\right)\right)]\delta_{c_w} \end{aligned}$$

And thus

$$\alpha\left(\gamma L + (1-\gamma) L'\right) = \gamma\alpha(L) + (1-\gamma)\alpha\left(L'\right).$$

Thus we can take $U(L) = \alpha(L)$ and $u_n = \alpha(\delta_{c_n})$.

Proof (cont.)

iii) Finally, let us show that if u' and u'' are the Bernoull utility functions representing $\succeq$, then $u'' = \beta u' + \gamma$ where $\beta > 0$. Let U' and U'' be the corresponding von Neumann-Morgenstern utility functions. The claim is proved if we prove it for U' and U''.
As before, let $\alpha'(L)$ solve

$$U'(L) = \alpha'(L)\, U'(\delta_{c_b}) + \left(1 - \alpha'(L)\right) U'(\delta_{c_w}).$$

Thus

$$\alpha'(L) = \frac{U'(L) - U'(\delta_{c_w})}{U'(\delta_{c_b}) - U'(\delta_{c_w})}.$$

But now since U'' is also a representation, we have

$$U''(L) = \alpha'(L)\, U''(\delta_{c_b}) + \left(1 - \alpha'(L)\right) U''(\delta_{c_w}).$$

Proof (cont.)

Plugging in the value of $\alpha'(L)$ and rearranging, we get:

$$U''(L) = \beta U'(L) + \gamma,$$

where

$$\beta = \frac{U''(\delta_{c_b}) - U''(\delta_{c_w})}{U'(\delta_{c_b}) - U'(\delta_{c_w})}$$

and

$$\gamma = U''(\delta_{c_w}) - U'(\delta_{c_w}) \frac{U''(\delta_{c_b}) - U''(\delta_{c_w})}{U'(\delta_{c_b}) - U'(\delta_{c_w})}.$$

Observations:

- Expected utility representation is a huge simplification for use in descriptive applications.
- Can be put to use in normative exercises: i.e. construct more complicated preferences from simple preferences.
- Can be generalized to infinite C (with some modifications)

Problem: Expected Utility theory fails in many experimens

- The most famous example: Allais' Paradox shows violations of Independence Axiom.
- Possible responses:
  - Normative response: one should be ready to correct "mistaken" behavior
  - Allais paradox is not very relevant for economics, since it applies only in special circumstances (large payoffs, probabilities close to 0 and 1)
  - Define preferences more broadly (e.g. regret theory)
  - Give up independence axiom to accommodate this behavior

Subjective Probability Theory (Savage, 1954)

This is just a quick glance on the topic (see Kreps, 1988, for more details).
Building blocks:

- Consequences as before, C.
- State of the World: Complete list of all relevant data for the problem at hand, Ω.
- Acts: Functions $f : \Omega \to C$.
- Preferences are defined on $\mathcal{F}$, the set of all possible acts.

- The idea is to deduce from the preferences the following:
  i) A subjective probability assessment $p(\omega)$ on Ω.
  ii) A utility function on $u : C \to \mathbb{R}$.
  Such that for all $f, f' \in \mathcal{F}$,

$$f \succsim f' \Leftrightarrow \int u(f(\omega))\, dp(\omega) \geq \int u\left(f'(\omega)\right) dp(\omega).$$

  or

$$f \succsim f' \Leftrightarrow \sum_{\omega \in \Omega} u(f(\omega))\, p(\omega) \geq \sum_{\omega \in \Omega} u\left(f'(\omega)\right) p(\omega).$$

- Notice that the end result looks similar to von Neumann-Morgenstern theory. However, underlying philosophy is different.
- In principle, assuming a set of Axioms holds, one can recover both, beliefs and utility funtions, with sufficient data
- Since also probabilities must be accounted, more axioms needed than under vNM. As a consequence, characterization complex (see Kreps, 1988)

Anscombe-Aumann (1963) assume that acts are

$$f : \Omega \to \Delta(C),$$

or in words, acts assign objective lotteries to states of the world.
This combines subjective probabilities and objective probabilities
Technically simpler than the fully subjective view.

State-dependent utility

Sometimes in applications it is convenient to use state-dependent utility functions:

$$u : \Omega \times C \rightarrow \mathbb{R}.$$

This is a convenient device to handle situations where C is directly chosen by the decision maker, and ω is chosen exogenously "by nature".
Expected utility formula in this framework is:

$$U(c) = \sum_{\omega \in \Omega} u(\omega, c)\, p(\omega).$$

Monetary consequences

- For the rest of the lecture (and the next), consequences are monetary amounts.
- $x \in \mathbb{R}$ is the final wealth of the decision maker.
- Analyze different $u : \mathbb{R} \rightarrow \mathbb{R}$.
- $F(x)$ denotes the distribution function of a monetary lottery.
- Discrete or continuous.

- Expected Utility theorem for a general distribution $F(x)$:

$$U(F) = \int_{x \in \mathbb{R}} u(x) dF(x). \tag{1}$$

- This encompasses both discrete distributions:

$$U(F) = \sum_x u(x) p(x), \tag{2}$$

and continuous distributions

$$U(F) = \int_{x \in \mathbb{R}} u(x) f(x)\, dx$$

Observations:

- Note the distinction between U and u. Call the former "von-Neumann-Morgenstern" and the latter "Bernoulli" utility function.
- There are some technicalities in extending Expected Utility Theorem to infinite C
- E.g.: when do the integrals and sums above converge? (Cf. St.Petersburg's paradox)
- Note the symmetry in the formula between $u(x)$ and $F(x)$.
- When analyzing expected utility, we can consider variations in each of these two components.
  1. Risk attitudes: Fix $F(x)$ and compare different $u(x)$.
  2. Riskiness of lotteries: Fix $u(x)$ and compare different $F(x)$.

Risk attitudes

Definition

The certainty equivalent $c(F, u)$ of a lottery $F(X)$ for a decision maker with (Bernoulli) utility function defined by

$$u(c(F, u)) = \int u(x) dF(x) \tag{3}$$

We can discuss attitudes towards risk by comparing the certainty equivalents of a fixed lottery under different utility functions.

Risk attitudes

Definition

A decision maker with a utility function u is said to be risk averse if for all $F(x)$,

$$c(F,u) \leq \int x dF(x). \tag{4}$$

It is easy to prove that

Proposition

Utility function u is risk averse if and only if it is concave.

Risk loving attitudes are defined with the opposite inequalities.

Can second derivatives be used to measure risk aversion?

Definition

The Arrow-Pratt measure of absolute risk aversion, $r_A(x,u)$ of utility function u at wealth level x is given by:

$$r_A(x,u) = -\frac{u''(x)}{u'(x)}. \tag{5}$$

The following theorem shows that $r_A(x,u)$ is a good measure of risk aversion.

Proposition

The following are equivalent:
i) $r_A(x,u_2) \geq r_A(x,u_1)$ *for all* x.
ii) $c(F,u_2) \leq c(F,u_1)$ *for all* $F(x)$.
iii) There is a concave function $\psi(\cdot)$ *such that* $u_2(x) = \psi(u_1(x))$.

Comparisons across wealth levels

Definition

$u(\cdot)$ exhibits decreasing absolute risk aversion (DARA) if $r_A(x, u)$ is a decreasing function of x.

If u exhibits DARA, then the decision maker becomes less sensitive to risk as her wealth increases, i.e. she is willing to pay less to get rid of a risk.

Proposition

The following are equivalent:
i) u exhibits DARA.
ii) If $x_2 < x_1$, $u_2(z) = u(x_2 + z)$ is a concave transformation of $u_1(z) = u(x_1 + z)$.
iii) For any risk $F(z)$, we have $x - c(F, u_x)$ is decreasing in x, where $u_x(z) = u(x + z)$.

- A related concept is the measure of relative risk aversion:

$$r^R(x,u) = -\frac{xu''(x)}{u'(x)}.$$

- $r^R(x,u)$ measures the attitudes towards gambles proportional to wealth.
- (Weak) Empirical evidence suggests:
    - People are risk averse.
    - Absolute risk aversion decreases with wealth.
    - Relative risk aversion decreases or is constant with wealth.

Special types of utility functions:
Constant absolute risk aversion (CARA):

$$u(x) = -e^{-\gamma x}.$$

Then $r^A(x, u) = \gamma$ for all x.

Constant relative risk aversion (CRRA):

$$u(x) = \frac{x^{1-\rho}}{1-\rho} \text{ for } \rho \neq 1.$$

Then $r^R(s, u) = \rho$ for all x.
Show that in the limit as $\rho \to 1$, we get $u(x) = \ln x$.