

Microeconomic Theory I

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Lecture 4: Production

Overview of the contents

- Basic theory of the supply side of the markets
- ▣ Description of technology (Production set)
- Profit maximization problem
 - Exogenous: prices
 - Endogenous: output and input demands
- Cost minimization problem
- Comparative statics: involves only substitution effects
- First look at aggregate behavior and fundamental theorems of welfare economics
- Main material for this lecture: MWG Ch. 5

Primitives

- 1 Commodity Space $X = \mathbb{R}^L$
In contrast to Consumer theory, negative numbers possible:
Inputs: $y_i < 0$.
Outputs: $y_i > 0$.
- 2 Production Set
Summary of the technologically feasible input-output combinations
A subset $Y \subset \mathbb{R}^L$.
- 3 Behavioral Assumption:
Maximize profit in Y .
With linear prices, profit is $p \cdot y$.

Assumptions on production set

This is a list of possible properties (these are adopted selectively):

- 1 Y is non-empty and closed.
- 2 $Y \cap \mathbb{R}_+^L = \{0\}$. (No free lunch, inaction possible).
- 3 $Y - \mathbb{R}_+^L \subset Y$. (Free disposal).
- 4 $y \in Y \setminus \{0\} \implies -y \notin Y$. (Irreversibility).
- 5 Returns to scale:
 - 1 (DRS) $y \in Y \implies \alpha y \in Y$ for all $\alpha \in [0, 1]$.
 - 2 (IRS) $y \in Y \implies \alpha y \in Y$ for all $\alpha \in [1, \infty)$.
 - 3 (CRS) $y \in Y \implies \alpha y \in Y$ for all α .
- 6 $Y + Y \subset Y$ (Free entry).
- 7 Y is convex.

Alternative ways of describing the technology set:

- In general, we can use a transformation function F
 $Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$.
- The set $\partial Y = \{y \in \mathbb{R}^L : F(y) = 0\}$ is called the transformation frontier.
- The slope of the level curve of $F(\cdot)$ is called the marginal rate of transformation:

$$MRT_{lk}(y) = \frac{\partial F(y) / \partial y_l}{\partial F(y) / \partial y_k}$$

- Single output, many input case:
 - q is the output.
 - z_k are the inputs.
 - Production function $q = f(z_1, \dots, z_{L-1})$
 - $Y = \{(-z_1, \dots, z_{L-1}, q) : q \leq f(z_1, \dots, z_{L-1}) \text{ and } z_k \geq 0 \text{ for all } k\}$.
 - Marginal rate of technical substitution:

$$MRTS_{lk}(z) = \frac{\partial f(z) / \partial z_l}{\partial f(z) / \partial z_k}$$

Profit Maximization Problem

Firm's Problem: (PMP)

$$\max_{y \in Y} p \cdot y.$$

or:

$$\max_y p \cdot y, \text{ s.t. } F(y) \leq 0$$

- Observe: no budget constraint
- ▣ When is the problem well posed (i.e. when does it have a solution)?
- Denote the value function to (PMP) by $\pi(p)$.
- $\pi(p)$ is called the profit function.
- Let $y(p)$ denote the set of optimal choices at price p .
- First order condition: If $y^* \in y(p)$, then $p = \lambda \nabla F(y^*)$ for some $\lambda > 0$.

Single output technology:

In the case of a single output, we have

$$\max_{z \in \mathbb{R}_+^K} pf(z) - w \cdot z.$$

the first order conditions are:

$$\frac{\partial f(z)}{\partial z_k} \leq \frac{w_k}{p}, \text{ for all } k$$

$$\frac{\partial f(z)}{\partial z_k} = \frac{w_k}{p} \text{ if } z_k > 0.$$

Properties of the profit function and supply

Assume that Y is closed and satisfies the free disposal property. Then:

- 1 $\pi(p)$ is homogenous of degree one.
- 2 $\pi(p)$ is convex.
- 3 If Y is convex, then $Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \in \mathbb{R}_{++}^L\}$.
- 4 $y(p)$ is homogenous of degree zero.
- 5 If Y is convex, then $y(p)$ is convex for all p . If Y is strictly convex then $y(p)$ is either empty or single valued.
- 6 If $y(p)$ is single valued at p , then $\pi(\cdot)$ is differentiable at p and $\nabla \pi(p) = y(p)$. (Hotelling's lemma).
- 7 If $y(\cdot)$ is a function and differentiable at p , then $Dy(p) = D^2\pi(p)$ is a symmetric and positive semidefinite matrix with $Dy(p)p = 0$

Law of supply

- From properties 2 and 6 we get immediately:

$$\frac{\partial y_i}{\partial p_i} \geq 0.$$

- Interpretation: If the price of an output increases, then the supply increases.
- Hence we get 'Law of Supply' very easily.
- Also: If the price of an input increases, the demand for the input decreases - 'Law of Input Demand'.

Law of supply via revealed profit approach:

For any $y, y' \in Y$, we know that if $y(p) = y$ and $y(p') = y'$ then

$$p \cdot y \geq p \cdot y' \text{ and}$$

$$p' \cdot y' \geq p' \cdot y.$$

Let

$$\Delta p = (p' - p) \text{ and } \Delta y = (y' - y).$$

Then we get the Law of Supply immediately:

$$\Delta p \cdot \Delta y = (p' \cdot y' - p' \cdot y) + (p \cdot y - p \cdot y') \geq 0.$$

Cost Minimization

- For simplicity, assume a single output.
- For each quantity of output, q , find the least cost input combination that yields q .
- Denote input prices by $w_k > 0$.
- The problem is:

$$\min_{z \in \mathbb{R}_+^K} w \cdot z$$

$$\text{s.t. } q = f(z_1, \dots, z_K).$$

- The solutions to this problem are $z(w, q)$, the conditional factor demands.
- The value function is the cost function, $c(w, q)$

$$c(w, q) = w \cdot z(w, q).$$

Analogy between firm's cost minimization and consumer's expenditure minimization

- $z(w, q)$ is analogous to $h(p, u)$ in consumer theory and $c(w, q)$ is analogous to $e(p, u)$.
- So, interpret production function as utility function, and properties of $c(w, q)$ and $z(w, q)$ follow directly from those of $e(p, u)$ and $h(p, u)$
- But note, there is a big difference:
 - Preference representation u is unique only up to increasing transformations.
 - Production function f is a unique description of technology.
 - Hence properties such as concavity of f have real meaning, contrary to those of u

Properties of cost function and conditional factor demands

Assume a single output and that Y is closed and satisfies the free disposal property. Then:

- 1 $c(\cdot)$ is homogenous of degree 1 in w and nondecreasing in q
- 2 $c(\cdot)$ is concave in w
- 3 if $\{z \geq 0 : f(z) \geq q\}$ is convex $\forall q$, then $Y = \{(-z, q) : w \cdot z \geq c(w, q), \forall w \in \mathbb{R}_{++}^{L-1}\}$
- 4 $z(w, q)$ is homogenous of degree 0 in w
- 5 if $\{z \geq 0 : f(z) \geq q\}$ is convex, then $z(w, q)$ is a convex set; if $\{z \geq 0 : f(z) \geq q\}$ is strictly convex, then $z(w, q)$ is a function
- 6 if $z(w, q)$ is a function, then $z(w, q)$ is differentiable at w and satisfies $\nabla_w c(w, q) = z(w, q)$ (Shepard's lemma)
- 7 if $z(w, q)$ is differentiable at w , then $D_w z(w, q) = D_w^2 c(w, q)$ is symmetric and negative semidefinite with $D_w z(w, q) w = 0$
- 8 if $f(\cdot)$ is homogenous of degree 1, then $c(\cdot)$ and $z(\cdot)$ are homogenous of degree 1 in q
- 9 if $f(\cdot)$ is concave, then $c(\cdot)$ is convex in q .

Describing technologies:

- Marginal rate of technical substitution:

$$MRTS_{ij} = \frac{\frac{\partial f(z)}{\partial z_i}}{\frac{\partial f(z)}{\partial z_j}}$$

Slope of the Isoquant $\{z' \in \mathbb{R}_+^K : f(z') = q\}$ at z .

- Elasticity of substitution:

$$\sigma_{ij}(z) = \frac{d \ln \left(\frac{z_j}{z_i} \right)}{d \ln (MRTS_{ij})} = \frac{d \ln \left(\frac{z_j}{z_i} \right)}{d \ln \left(\frac{\frac{\partial f(z)}{\partial z_i}}{\frac{\partial f(z)}{\partial z_j}} \right)} = \frac{d \left(\frac{z_j}{z_i} \right)}{\left(\frac{z_j}{z_i} \right)} \frac{\frac{\partial f(z)}{\partial z_i} / \frac{\partial f(z)}{\partial z_j}}{d \left(\frac{\partial f(z)}{\partial z_i} / \frac{\partial f(z)}{\partial z_j} \right)}$$

- Elasticity of scale:

$$\mu(z) = \left[\frac{d[\ln f(tx)]}{d \ln(t)} \right]_{t=1} = \frac{\sum_{k=1}^K z_k \frac{\partial f(z)}{\partial z_k}}{f(z)}$$

- The Law of (aggregate) Supply follows:

$$\Delta p \cdot \Delta y \geq 0$$

- Let Y be the aggregate production set:

$$Y = Y_1 + \dots + Y_J = \{y \in \mathbb{R}^L : y = \sum_j y_j, \text{ for some } y_j \in Y_j, j = 1, \dots, J\}$$

- Let $\pi^*(p), y^*(p)$ be the corresponding profits and supply correspondences.

Output choice

- Using the cost function derived above, PMP can be restated as the problem of choosing the optimal level of production:

$$\max_{q \in \mathbb{R}} pq - c(w, q)$$

- Hence we get the FOC:

$$p = \frac{\partial c(w, q)}{\partial q}$$

- For competitive firms, marginal cost equals price.
- Note: $\frac{\partial c(w, q)}{\partial q} = \lambda$ for the cost minimization problem.

Aggregation

- Since there are only substitution effects along the production frontier, the aggregation theory for the supply side is simple
- Y_1, \dots, Y_J , collection of production sets
- $\pi_j(p), y_j(p)$, profits and supply correspondences, $j = 1, \dots, J$
- the aggregate supply is

$$y(p) = \sum_j y_j(p) = \{y \in \mathbb{R}^L : y = \sum_j y_j, \text{ for some } y_j \in y_j(p), j = 1, \dots, J\}$$

- The properties of $y_j(p)$ are preserved under addition. In particular, $Dy(p) = D^2 \pi(p)$ is a symmetric and positive semidefinite.

Theorem

For all $p \gg 0$, we have:

- 1 $\pi^*(p) = \sum_j \pi_j(p)$,
- 2 $y^*(p) = \sum_j y_j(p)$.

Proof is easy. Note that similar aggregation results are not possible on the demand side (see MWG chapter 4).

Definition

A production vector y is efficient if there is no $y' \in Y$ s.t. $y' \geq y$ and $y' \neq y$.

Proposition

If $y \in y^*(p)$ for some $p \in \mathbb{R}_{++}^L$, then y is efficient.

- Compare to first fundamental theorem of welfare economics

Proposition

If Y is convex, then every efficient $y \in Y$ satisfies $y \in y^*(p)$ for some $p \in \mathbb{R}_+^L$.

- Compare: second fundamental theorem of welfare economics
- Note: in the first result $p \in \mathbb{R}_{++}^L$ whereas in the second $p \in \mathbb{R}_+^L$.

Discussion

Differences between Consumer and Producer Theory:

- Preference representation u is unique only up to increasing transformations, whereas firm's profit has a real meaning
- On the other hand, while it is natural to take preferences as the primitive defining consumer behavior, is it clear that profit maximization should be the firm's objective?
 - Since increased profits increase owners' wealth, profit maximization is indeed well justified under certain conditions
- There is no budget constraint in the firm's problem
 - Wealth effects are missing, so as you have seen, things are much more simple on production side