Microeconomic Theory I

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Lecture 4: Production

Overview of the contents

- Basic theory of the supply side of the markets
- Description of technology (Production set)
- Profit maximization problem
  - Exogenous: prices
  - Endogenous: output and input demands
- Cost minimization problem
- Comparative statics: involves only substitution effects
- First look at aggregate behavior and fundamental theorems of welfare economics
- Main material for this lecture: MWG Ch. 5

Primitives

1. Commodity Space $X = \mathbb{R}^L$
   In contrast to Consumer theory, negative numbers possible:
   Inputs: $y_i < 0$.
   Outputs: $y_i > 0$.
2. Production Set
   Summary of the technologically feasibe input-output combinations
   A subset $Y \subset \mathbb{R}^L$.
3. Behavioral Assumption:
   Maximize profit in Y.
   With linear prices, profit is $p \cdot y$.

Assumptions on production set

This is a list of possible properties (these are adopted selectively):

1. Y is non-empty and closed.
2. $Y \cap \mathbb{R}^L_+ = \{0\}$. (No free lunch, inaction possible).
3. $Y - \mathbb{R}^L_+ \subset Y$. (Free disposal).
4. $y \in Y \backslash \{0\} \implies -y \notin Y$. (Irreversibility).
5. Returns to scale:
    1. (DRS) $y \in Y \implies \alpha y \in Y$ for all $\alpha \in [0, 1]$.
    2. (IRS) $y \in Y \implies \alpha y \in Y$ for all $\alpha \in [1, \infty)$.
    3. (CRS) $y \in Y \implies \alpha y \in Y$ for all α.
6. $Y + Y \subset Y$ (Free entry).
7. Y is convex.

Alternative ways of describing the technology set:

- In general, we can use a transformation function F

$$Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}.$$

  - The set $\partial Y = \{y \in \mathbb{R}^L : F(y) = 0\}$ is called the transformation frontier.
  - The slope of the level curve of $F(\cdot)$ is called the marginal rate of transformation:

$$MRT_{lk}(y) = \frac{\partial F(y)/\partial y_l}{\partial F(y)/\partial y_k}$$

- Single output, many input case:
  - q is the output.
  - z_k are the inputs.
  - Production function $q = f(z_1, ..., z_{L-1})$
  - $Y = \{(-z_1, ..., z_{L-1}, q) : \ q \leq f(z_1, ..., z_{L-1}) \text{ and } \ z_k \geq 0 \text{ for all } k\}.$
  - Marginal rate of technical substitution:

$$MRTS_{lk}(z) = \frac{\partial f(z)/\partial z_l}{\partial f(z)/\partial z_k}$$

Profit Maximization Problem

Firm's Problem: (PMP)

$$\max_{y \in Y} \; p \cdot y.$$

or:

$$\max_{y} \; p \cdot y, \text{ s.t. } F(y) \leq 0$$

- Observe: no budget constraint
- When is the problem well posed (i.e. when does it have a solution)?
- Denote the value function to (PMP) by $\pi(p)$.
- $\pi(p)$ is called the profit function.
- Let $y(p)$ denote the set of optimal choices at price p.
- First order condition: If $y^* \in y(p)$, then $p = \lambda \nabla F(y^*)$ for some $\lambda > 0$.

Single output technology:

In the case of a single output, we have

$$\max_{z \in \mathbb{R}_+^K} \quad pf(z) - w \cdot z.$$

the first order conditions are:

$$\begin{aligned} \frac{\partial f(z)}{\partial z_k} &\leq \frac{w_k}{p}, \text{ for all } k \\ \frac{\partial f(z)}{\partial z_k} &= \frac{w_k}{p} \text{ if } z_k > 0. \end{aligned}$$

Properties of the profit function and supply

Assume that Y is closed and satisfies the free disposal property. Then:

1. $\pi(p)$ is homogenous of degree one.
2. $\pi(p)$ is convex.
3. If Y is convex, then $Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \in \mathbb{R}^L_{++}\}$.
4. $y(p)$ is homogenous of degree zero.
5. If Y is convex, then $y(p)$ is convex for all p. If Y is strictly convex then $y(p)$ is either empty or single valued.
6. If $y(p)$ is single valued at p, then $\pi(\cdot)$ is differentiable at p and $\nabla\pi(p) = y(p)$. (Hotelling's lemma).
7. If $y(\cdot)$ is a function and differentiable at p, then $Dy(p) = D^2\pi(p)$ is a symmetric and positive semidefinite matrix with $Dy(p)p = 0$

Law of supply

- From properties 2 and 6 we get immediately:

$$\frac{\partial y_i}{\partial p_i} \geq 0.$$

- Interpretation: If the price of an output increases, then the supply increases.
- Hence we get 'Law of Supply' very easily.
- Also: If the price of an input increases, the demand for the input decreases - 'Law of Input Demand'.

Law of supply via revealed profit approach:

For any $y, y' \in Y$, we know that if $y(p) = y$ and $y(p') = y'$ then

$$p \cdot y \geq p \cdot y' \text{ and}$$

$$p' \cdot y' \geq p' \cdot y.$$

Let

$$\Delta p = \left(p' - p\right) \text{ and } \Delta y = \left(y' - y\right).$$

Then we get the Law of Supply immediately:

$$\Delta p \cdot \Delta y = \left(p' \cdot y' - p' \cdot y\right) + \left(p \cdot y - p \cdot y'\right) \geq 0.$$

Cost Minimization

- For simplicity, assume a single output.
- For each quantity of output, q, find the least cost input combination that yields q.
- Denote input prices by $w_k > 0$.
- The problem is:

$$\min_{z \in \mathbb{R}_+^K} \quad w \cdot z$$
$$\text{s.t. } q = f\left(z_1, ..., z_K\right).$$

- The solutions to this problem are $z\left(w, q\right)$, the conditional factor demands.
- The value function is the cost function, $c\left(w, q\right)$

$$c\left(w, q\right) = w \cdot z\left(w, q\right).$$

Analogy between firm's cost minimization and consumer's expenditure minimization

- $z(w, q)$ is analogous to $h(p, u)$ in consumer theory and $c(w, q)$ is analogous to $e(p, u)$.
- So, interprete production function as utility function, and properties of $c(w, q)$ and $z(w, q)$ follow directly from those of $e(p, u)$ and $h(p, u)$
- But note, there is a big difference:
    - Preference representation u is unique only up to increasing transformations.
    - Production function f is a unique description of technology.
    - Hence properties such as concavity of f have real meaning, contrary to those of u

Properties of cost function and conditional factor demands

Assume a single output and that Y is closed and satisfies the free disposal property. Then:

1. $c()$ is homogenous of degree 1 in w and nondecreasing in q
2. $c()$ is concave in w
3. if $\{z \geq 0 : f(z) \geq q\}$ is convex $\forall q$, then $Y = \{(-z, q) : w \cdot z \geq c(w, q), \forall w \in \mathbb{R}_{++}^{L-1}\}$
4. $z(w, q)$ is homogenous of degree 0 in w
5. if $\{z \geq 0 : f(z) \geq q\}$ is convex , then $z(w, q)$ is a convex set; if $\{z \geq 0 : f(z) \geq q\}$ is strictly convex, then $z(w, q)$ is a function
6. if $z(w, q)$ is a function, then $z(w, q)$ is differentiable at w and satisfies $\nabla_w c(w, q) = z(w, q)$ (Shepard's lemma)
7. if $z(w, q)$ is diffentiable at w, then $D_w z(w, q) = D_w^2 c(w, q)$ is symmetric and negative semidefinite with $D_w z(w, q) w = 0$
8. if $f()$ is homogenous of degree 1, then $c()$ and $z()$ are homogenous of degree 1 in q
9. if $f()$ is concave, then $c()$ is convex in q.

Output choice

- Using the cost function derived above, PMP can be restated as the problem of choosing the optimal level of production:

$$\max_{q\in\mathbb{R}} \; pq - c\left(w, q\right).$$

- Hence we get the FOC:

$$p = \frac{\partial c\left(w, q\right)}{\partial q}.$$

- For competitive firms, marginal cost equals price.
- Note: $\frac{\partial c(w,q)}{\partial q} = \lambda$ for the cost minimization problem.

Describing technologies:

- Marginal rate of technical substitution:

$$MRTS_{ij} = \frac{\frac{\partial f(z)}{\partial z_i}}{\frac{\partial f(z)}{\partial z_j}}.$$

Slope of the Isoquant $\left\{z' \in \mathbb{R}_+^K : f\left(z'\right) = q\right\}$ at z.

- Elasticity of substitution:

$$\sigma_{ij}(z) = \frac{d \ln\left(\frac{z_j}{z_i}\right)}{d \ln\left(MRTS_{ij}\right)} = \frac{d \ln\left(\frac{z_j}{z_i}\right)}{d \ln\left(\frac{\frac{\partial f(z)}{\partial z_i}}{\frac{\partial f(z)}{\partial z_j}}\right)} = \frac{d\left(\frac{z_j}{z_i}\right)}{\left(\frac{z_j}{z_i}\right)} \frac{\frac{\partial f(z)}{\partial z_i} / \frac{\partial f(z)}{\partial z_j}}{d\left(\frac{\partial f(z)}{\partial z_i} / \frac{\partial f(z)}{\partial z_j}\right)}.$$

- Elasticity of scale:

$$\mu(z) = \left[\frac{d\left[\ln f(tx)\right]}{d \ln(t)}\right]_{t=1} = \frac{\sum_{k=1}^K z_k \frac{\partial f(z)}{\partial z_k}}{f(z)}.$$

Aggregation

- Since there are only substiution effects along the production frontier, the aggregation theory for the supply side is simple
- $Y_{1,}...., Y_J$, collection of production sets
- $\pi_j(p)$, $y_j(p)$, profits and supply correspondences, $j = 1, ..., J$
- the aggregate supply is

$$y(p) = \sum_j y_j(p) = \{y \in \mathbb{R}^L : y = \sum_j y_j, \text{ for some } y_j \in y_j(p), j = 1, ..,$$

- The properties of $y_j(p)$ are preserved under addition. In particular, $Dy(p) = D^2\pi(p)$ is a symmetric and positive semidefinite.

- The Law of (aggregate) Supply follows:

$$\Delta p \cdot \Delta y \geq 0$$

- Let Y be the aggregate production set:

$$Y = Y_1 + ... + Y_J = \{y \in \mathbb{R}^L : y = \sum_j y_j, \text{ for some } y_j \in Y_j, j = 1, .., J\}$$

- Let $\pi^* (p)$, $y^* (p)$ be the corresponding profits and supply correspondences.

Theorem

For all $p \gg 0$, we have:

1. $\pi^* (p) = \sum_j \pi_j (p)$,
2. $y^* (p) = \sum_j y_j (p)$.

Proof is easy. Note that similar aggregation results are not possible on the demand side (see MWG chapter 4).

Efficiency

Definition

A production vector y is efficient if there is no $y' \in Y$ s.t. $y' \geq y$ and $y' \neq y$.

Proposition

If $y \in y^*(p)$ for some $p \in \mathbb{R}^L_{++}$, then y is efficient.

- Compare to first fundamental theorem of welfare economics

Efficiency

Proposition

If Y is convex, then every efficien $y \in Y$ satisfies $y \in y^*(p)$ for some $p \in \mathbb{R}^L_+$.

- Compare: second fundamental theorem of welfare economics
- Note: in the first result $p \in \mathbb{R}^L_{++}$ whereas in the second $p \in \mathbb{R}^L_+$.

Discussion

Differences between Consumer and Producer Theory:

- Preference representation u is unique only up to increasing transformations, whereas firm's profit has a real meaning
- On the othet hand, while it is natural to take preferences as the primitive defining consumer behavior, is it clear that profit maximization should be the firm's objective?
  - Since increased profits increase owners' wealth, profit maximization is indeed well justified under certain conditions
- There is no budget constraint in the firm's problem
  - Wealth effects are missing, so as you have seen, things are much more simple on production side