

Microeconomic Theory I

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Lecture 3: Classical Demand Theory

Overview of the contents

- Preference-based approach to consumer theory
- Dual relationship between utility maximization and expenditure minimization problems
- Integrability
- Welfare analysis
- Main material for this lecture: MWG Ch. 3
- Additional material: Deaton and Muellbauer (1980)

Framework:

- X the choice set, the consumption set
- We take $X = \mathbb{R}_+^L$, where $L \in \mathbb{N}$.
- $x = (x_1, \dots, x_L)$, where each $x_l \in \mathbb{R}_+$ for each $l \in \{1, \dots, L\}$.
- $B(p, w) = \{x \in X \mid p \cdot x \leq w\}$.
- $p \gg 0$
- Local non-satiation: implies Walras' Law.
- Strict convexity of preferences: implies that $x(p, w)$ is single valued.
- Continuous preferences: implies the existence of a continuous representation (does not say that all representations are continuous)

Utility Maximization Problem (UMP):

- Assume rational, continuous, and locally nonsatiated preferences
- Let $u(\cdot)$ be a continuous utility function representing these
- The consumer's problem is to choose her consumption bundle in order to maximize utility:

$$\begin{aligned} \max_{x \geq 0} & u(x) \\ \text{s.t.} & p \cdot x \leq w. \end{aligned}$$

Existence of a solution

Proposition

(Weierstrass). Let $f : X \rightarrow \mathbb{R}$ be a continuous function and X a compact set. Then f attains its maximum on x , i.e. there is a point $x^* \in X$ such that for all $x \in X$,

$$f(x) \leq f(x^*).$$

- Recall: if $X \subset \mathbb{R}^n$, then X is compact if and only if it is closed and bounded.
- It is easy to see through counterexamples that continuity, boundedness and closedness are all required for the result.
- How to characterize x^* ? Assume that $u(x)$ is differentiable and use Kuhn-Tucker theory.

Properties of the solution

We already discussed the following properties:

Theorem

Suppose $u(\cdot)$ is a continuous representation of a locally nonsatiated preference relation defined on \mathbb{R}_+^L . Then the Walrasian demand correspondence $x(p, w)$ possesses the following properties:

- Homogeneity of degree zero in (p, w)
- Walras' law: $p \cdot x = w$ for all $x \in x(p, w)$
- Convexity/uniqueness: If preferences are convex, then $x(p, w)$ is a convex set. If preferences are strictly convex, then $x(p, w)$ is a singleton

Check that you know how to prove these.

Kuhn-Tucker necessary conditions

There exists a scalar $\lambda \in \mathbb{R}$ such that for all l :

$$\begin{aligned} \frac{\partial u(x^*)}{\partial x_l} - \lambda p_l &\leq 0, \text{ with equality if } x_l^* > 0, \\ \lambda (p \cdot x^* - w) &= 0, \\ x^* &\geq 0. \end{aligned}$$

Using vector notation:

$$\nabla u(x^*) \leq \lambda p \text{ and } x^* \cdot [\nabla u(x^*) - \lambda p] = 0.$$

These conditions are also sufficient if $u(\cdot)$ is quasiconcave and monotone, and if $\nabla u(x) \neq 0$ for all $x \in \mathbb{R}_+^L$.

Interpretation

Marginal rate of substitution (MRS) between goods l and k :

For all l, k such that $x_l^*, x_k^* > 0$, we have:

$$\frac{\frac{\partial u(x^*)}{\partial x_l}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_l}{p_k}.$$

Lagrange multiplier λ : the marginal utility value of wealth

$$\lambda = D_w u(x(p, w)).$$

To see this, use chain rule to get

$$D_w u(x(p, w)) = \nabla u(x) \cdot D_w x(p, w).$$

FOC's give $\nabla u(x) = \lambda p$ and Engel aggregation gives $p \cdot D_w x(p, w) = 1$.

Alternatively, use directly the Envelope Theorem:

Let $z(\bar{q})$ denote value function depending on a parameter vector \bar{q}

Objective is $f(x; \bar{q})$ and $g_1(x(\bar{q}); \bar{q}), \dots, g_m(x(\bar{q}); \bar{q})$ are constraints.

The Envelope Theorem for the constrained optimization problem says that

$$\nabla z(\bar{q}) = \nabla_q f(x(\bar{q}); \bar{q}) - \sum_{m=1}^M \lambda_m \nabla_q g_m(x(\bar{q}); \bar{q})$$

For the UMP, this equation gives the interpretation for λ discussed above.

Indirect Utility Function

The value of UMP is called the indirect utility function:

$$v(p, w) \equiv u(x(p, w)).$$

What are the properties of $v(p, w)$ implied by the utility maximization problem?

Properties of the indirect utility function

Theorem

Suppose $u(\cdot)$ is a continuous representation of a locally nonsatiated preference relation defined on \mathbb{R}_+^L . The indirect utility function $v(p, w)$ is

- Homogeneous of degree zero
- Strictly increasing in w and nonincreasing in p_l for any l
- Quasiconvex
- Continuous in p and w

Proofs are straightforward (discussed in the class)

Deriving Walrasian demand from indirect utility function

- Given $x(p, w)$, indirect utility function is simply $v(p, w) = u(x(p, w))$
- In the other direction, we have Roy's Identity:

Theorem

Suppose $u(\cdot)$ is a continuous representation of a locally nonsatiated and strictly convex preference relation defined on \mathbb{R}_+^L . Suppose further that $v(\cdot, \cdot)$ is differentiable at $(\bar{p}, \bar{w}) \gg 0$. Then:

$$x(\bar{p}, \bar{w}) = - \frac{\nabla_p v(\bar{p}, \bar{w})}{\nabla_w v(\bar{p}, \bar{w})}.$$

Proof e.g. by envelope theorem.

Expenditure Minimization Problem (EMP)

Consider next the related problem of expenditure minimization (EMP).

$$\begin{aligned} \min_{x \in \mathbb{R}_+^L} \quad & p \cdot x \\ \text{s.t.} \quad & u(x) \geq u. \end{aligned}$$

Notice that even though the feasible set is not bounded, the problem has a solution when $p \in \mathbb{R}_{++}^L$. (we assume this throughout).

Hicksian demand and expenditure function

- The solution to EMP, denoted by $h(p, u)$, is called the Hicksian or compensated demand function
- The value of EMP, denoted by $e(p, u)$, is called the expenditure function
- The following observation is crucial:

Proposition

Fix a price vector $p \in \mathbb{R}_{++}^L$ and a continuous utility function $u(x)$ representing locally non-satiated preference relation.

- If $x^* = x(p, w)$, then $x^* = h(p, u(x^*)) = h(p, v(p, w))$.
- If $x^* = h(p, u)$, then $x^* = x(p, p \cdot x^*) = x(p, e(p, u))$.

Summarizing:

$$x(p, w) = h(p, v(p, w)) \text{ and } h(p, u) = x(p, e(p, u)).$$

Obviously then also:

$$w = e(p, v(p, w)) \text{ and } u = v(p, e(p, u)).$$

Properties of the expenditure function

It is straightforward to derive the following properties:

Theorem

Suppose $u(\cdot)$ is a continuous representation of a locally nonsatiated preference relation defined on \mathbb{R}_+^L . The expenditure function $e(p, u)$ is

- Homogeneous of degree one in p
- Strictly increasing in u and nondecreasing in p_l for any l
- Concave in p
- Continuous in p and u

Properties of the Hicksian demand function

Theorem

Suppose $u(\cdot)$ is a continuous representation of a locally nonsatiated preference relation defined on \mathbb{R}_+^L . Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ possesses the following properties:

- Homogeneity of degree zero in p
- No excess utility: For any $x \in h(p, u)$, $u(x) = u$
- Convexity/uniqueness: If preferences are convex, then $h(p, u)$ is a convex set. If preferences are strictly convex, then $h(p, u)$ is a singleton.

Deriving Hicksian demand from expenditure function

- Given Hicksian demand $h(p, u)$, the expenditure function is simply $e(p, u) = p \cdot h(p, u)$
- In the other direction, we have:

Theorem

Suppose $u(\cdot)$ is a continuous representation of a locally nonsatiated and strictly convex preference relation defined on \mathbb{R}_+^L . For all p and u , we have:

$$h(p, u) = \nabla_p e(p, u).$$

Proof similar to Roy's identity.

Price derivatives of the Hicksian demand

- Denote by $D_p h(p, u)$ the $L \times L$ matrix of derivatives of $h(p, u)$ w.r.t. prices
- Given continuous, nonsatiated, and strictly convex preferences represented by a continuous $u(x)$, the derivative matrix $D_p h(p, u)$ is negative semidefinite, symmetric, and satisfies $D_p h(p, u)p = 0$.
- Negative semidefiniteness is the compensated law of demand:

$$dp \cdot D_p h(p, u) dp \leq 0 \text{ for all } dp, \text{ that is } dp \cdot dh \leq 0.$$

- Note that the compensation is different from the previous lecture:
 - Hicksian demand gives optimal consumption when wealth is adjusted in order to keep utility level constant
 - In the previous lecture, we had a different compensation: change wealth in order to maintain original consumption bundle just affordable

Slutsky Equation

- Hicksian demand is not directly observable, whereas Walrasian demand in principle is
- Nevertheless, price derivative of Hicksian demand can be derived from Walrasian demand:

Theorem

Suppose $u(\cdot)$ is a continuous representation of a locally nonsatiated and strictly convex preference relation defined on \mathbb{R}_+^L . Then, for all (p, w) and $u = v(p, w)$, we have

$$D_p h(p, u) = D_p x(p, w) + D_w x(p, w) x(p, w)^T$$

Some comments

- Slutsky equation translates properties of $D_p h(p, u)$ into restrictions on Walrasian demand (in particular, the negative semidefiniteness of $D_p h(p, u)$)
- Recall the properties of $S(p, w)$, the substitution matrix defined in the choice-based approach
- $D_p h(p, u)$ is symmetric but $S(p, w)$ need not be.
- The restrictions under the preference-based approach are stronger, so we obtain additional observable implications.

Integrability

- utility maximization implies: (1) Walras' Law, (2) Homogeneity of degree 0 for the demands and, (3) $S(p, w)$ is symmetric and negative semidefinite (s.n.s.d.).
- can we also conclude that (1)-(3) imply that there exists a rational preference relation that could have generated $x(p, w)$
- the integrability problem: the answer is yes
- (1)-(3) are the only consequences of preference maximization
- The route we follow: $x(p, w) \rightarrow e(p, u) \rightarrow \succeq$

- consider first $x(p, w) \rightarrow e(p, u)$
- recall $x(p, w) = x(p, e(p, u)) = h(p, u)$, so

$$\nabla_p e(p, u) = x(p, e(p, u))$$

Note:

- a system of PDE with $e(p^0, u) = w^0$ as an initial condition. Solved for e as a function of p along a given indifference surface
- if there exists a solution $e(p) = e(p, u)$, then we must have

$$D_p^2 e(p) = S(p, e(p))$$

- where $S(p, e(p))$ is symmetric, negative semidefinite (s.n.s.d.) Slutsky evaluated along the solution
- s.n.s.d. Slutsky is also sufficient condition by the Frobenius' Theorem

- consider then $e(p, u) \rightarrow \succeq$
- Recall the properties of $e(p, u)$:
 - i) $e(p, u)$ is homogenous of degree 1 in p .
 - ii) Strictly increasing in u and non-decreasing in p_l for all l .
 - iii) Concave in p .
 - iv) Continuous in p, u .
- Again we have conversely:

Proposition

Suppose that $e(p, u)$ satisfies i)-iv). Then there exists a utility function $u: \mathbb{R}_+^L \rightarrow \mathbb{R}$ such that $e(p, u)$ is the expenditure function of the associated EMP. Furthermore, the upper level sets for this utility function are given by

$$V_u = \{x \in \mathbb{R}_+^L \mid p \cdot x \geq e(p, u) \text{ for all } p \in \mathbb{R}_+^L\}.$$

Then also, $e(p, u) = \min\{p \cdot x \mid x \in V_u\}$.

where we have set $w = e(p, u) = e(p, v(p, w))$.

Theorem

(Integrability)

Suppose that $x(p, w)$ is homogenous of degree 0, satisfies Walras' law, and has a symmetric and negative semi-definite substitution matrix. Then there is a (quasiconcave) utility function $u(x)$ such that $x(p, w)$ is the solution to UMP.

- How to quantify the welfare effect of a change in the economic environment?
- For simplicity, consider a change in the prices
- Remember that utility is an ordinal concept, so it is not possible to give an unambiguous measure of welfare
- However, if we know preferences, we can use any indirect utility function to rank different price-wealth situations
- Moreover, we can choose some indirect utility function that measures welfare changes in monetary units
- These are called money metric indirect utility functions

Money metric indirect utility function

- Start with any indirect utility function $v(p, w)$
- Choose any price vector $\bar{p} \gg 0$
- Then, function $e(\bar{p}, v(p, w))$ is also an indirect utility function
- It gives the wealth required to reach utility $v(p, w)$ at prices \bar{p}
- Thus, $e(\bar{p}, v(p^1, w)) - e(\bar{p}, v(p^0, w))$ is a measure of welfare due to price change from p^0 to p^1 expressed in monetary units
- Note that the particular choice of v does not make any difference - only the properties of the preferences and the choice of \bar{p} matter

- How to choose \bar{p} ?
- Two natural possibilities: $\bar{p} = p^0$ and $\bar{p} = p^1$
- The former is the equivalent variation (EV), the latter the compensated variation (CV)
- Formally, writing $u^0 = v(p^0, w)$ and $u^1 = v(p^1, w)$, we have:

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w$$

$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0)$$
- We have: $v(p^0, w + EV) = u^1$ and $v(p^1, w - CV) = u^0$

- We can also write EV and CV using Hicksian demand curve
- Suppose that only the price of good 1 changes ($p_l^0 = p_l^1 = \bar{p}_l$ for all $l \neq 1$). Then:

$$EV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^1) dp_1$$

$$CV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^0) dp_1$$

- We have $EV(p^0, p^1, w) > (<) CV(p^0, p^1, w)$ if good 1 is normal (inferior)
- What kind of preferences give $EV(p^0, p^1, w) = CV(p^0, p^1, w)$?