Microeconomic Theory I

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Lecture 3: Classical Demand Theory

Overview of the contents

- Preference-based approach to consumer theory
- Dual relationship between utility maximization and expenditure minimization problems
- Integrability
- Welfare analysis
- Main material for this lecture: MWG Ch. 3
- Additional material: Deaton and Muellbauer (1980)

Framework:

- X the choice set, the consumption set
- We take $X = \mathbb{R}^L_+$, where $L \in \mathbb{N}$.
- $x = (x_1, ..., x_L)$, where each $x_l \in \mathbb{R}_+$ for each $l \in \{1, ..., L\}$.
- $B(p, w) = \{x \in X \,|p \cdot x \leq w\}$.
- $p \gg 0$
- Local non-satiation: implies Walras' Law.
- Strict convexity of preferences: implies that $x(p, w)$ is single valued.
- Continuous preferences: implies the existence of a continuous representation (does not say that all representations are continuous)

Utility Maximization Problem (UMP):

- Assume rational, continuous, and locally nonsatiated preferences
- Let $u(\cdot)$ be a continuous utility function representing these
- The consumer's problem is to choose her consumption bundle in order to maximize utility:

$$\max_{x \geq 0} \; u(x)$$

$$\text{s.t. } p \cdot x \leq w.$$

Existence of a solution

Proposition

(Weierstrass). Let $f : X \to \mathbb{R}$ *be a continuous function and* X *a compact set. Then* f *attains its maximum on* x*, i.e. there is a point* $x^* \in X$ *such that for all* $x \in X$,

$$f(x) \leq f(x^*).$$

- Recall: if $X \subset \mathbb{R}^n$, then X is compact if and only if it is closed and bounded.
- It is easy to see through counterexamples that continuity, boundedness and closedness are all required for the result.
- How to characterize x^*? Assume that $u(x)$ is differentiable and use Kuhn-Tucker theory.

Properties of the solution

We already discussed the following properties:

Theorem

Suppose $u(\cdot)$ is a continuous representation of a locally nonsatiated preference relation defined on $\mathbb{R}^L_+$. Then the Walrasian demand correspondence $x(p,w)$ possesses the following properties:

- *Homogeneity of degree zero in (p,w)*
- *Walras' law: $p \cdot x = w$ for all $x \in x(p,w)$*
- *Convexity/uniqueness: If preferences are convex, then $x(p,w)$ is a convex set. If preferences are strictly convex, then $x(p,w)$ is a singleton*

Check that you know how to prove these.

Kuhn-Tucker necessary conditions

There exists a scalar $\lambda \in \mathbb{R}$ such that for all l:

$$\begin{aligned} \frac{\partial u\left(x^*\right)}{\partial x_l} - \lambda p_l &\leq 0, \text{ with equality if } x_l^* > 0, \\ \lambda\left(p \cdot x^* - w\right) &= 0, \\ x^* &\geq 0. \end{aligned}$$

Using vector notation:

$$\nabla u\left(x^*\right) \leq \lambda p \text{ and } x^* \cdot \left[\nabla u\left(x^*\right) - \lambda p\right] = 0.$$

These conditions are also sufficient if $u\left(\cdot\right)$ is quasiconcave and monotone, and if $\nabla u\left(x\right) \neq 0$ for all $x \in \mathbb{R}^L_+$.

Interpretation

Marginal rate of substitution (MRS) between goods l and k :
For all l, k such that $x_l^*, x_k^* > 0$, we have:

$$\frac{\frac{\partial u(x^*)}{\partial x_l}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_l}{p_k}.$$

Lagrange multiplier λ: the marginal utility value of wealth

$$\lambda = D_w u\left(x\left(p, w\right)\right).$$

To see this, use chain rule to get

$$D_w u\left(x\left(p, w\right)\right) = \nabla u\left(x\right) \cdot D_w x\left(p, w\right).$$

FOC's give $\nabla u\left(x\right) = \lambda p$ and Engel aggregation gives $p \cdot D_w x\left(p, w\right) = 1$.

Alternatively, use directly the Envelope Theorem:
Let $z(\bar{q})$ denote value function depending on a parameter vector $\bar{q}$
Objective is $f(x;\bar{q})$ and $g_1(x(\bar{q});\bar{q}), ...g_m(x(\bar{q});\bar{q})$ are constraints.
The Envelope Theorem for the constrained optimization problem says that

$$\nabla z(\bar{q}) = \nabla_q f(x(\bar{q});\bar{q}) - \sum_{m=1}^{M} \lambda_m \nabla_q g_m(x(\bar{q});\bar{q})$$

For the UMP, this equation gives the interpretation for λ discussed above.

Indirect Utility Function

The value of UMP is called the indirect utility function:

$$v(p, w) \equiv u(x(p, w)).$$

What are the properties of $v(p, w)$ implied by the utility maximization problem?

Properties of the indirect utility function

Theorem

Suppose $u(\cdot)$ *is a continuous representation of a locally nonsatiated preference relation defined on* $\mathbb{R}_+^L$. *The indirect utility function* $v(p, w)$ *is*

- *Homogeneous of degree zero*
- *Stirctly increasing in* w *and nonincreasing in* p_l *for any* l
- *Quasiconvex*
- *Continuous in* p *and* w

Proofs are straightforward (discussed in the class)

Deriving Walrasian demand from indirect utility function

- Given $x(p, w)$, indirect utility function is simply $v(p, w) = u(x(p, w))$
- In the other direction, we have Roy's Identity:

Theorem

Suppose $u(\cdot)$ is a continuous representation of a locally nonsatiated and strictly convex preference relation defined on $\mathbb{R}^L_+$. Suppose further that $v(\cdot, \cdot)$ is differentiable at $(\overline{p}, \overline{w}) \gg 0$. Then:

$$x(\overline{p}, \overline{w}) = -\frac{\nabla_p v(\overline{p}, \overline{w})}{\nabla_w v(\overline{p}, \overline{w})}.$$

Proof e.g. by envelope theorem.

Expenditure Minimization Problem (EMP)

Consider next the related problem of expenditure minimization (EMP).

$$\min_{x \in \mathbb{R}_{+}^{L}} \; p \cdot x$$

$$\text{s.t. } u\left(x\right) \geq u.$$

Notice that even though the feasible set is not bounded, the problem has a solution when $p \in \mathbb{R}_{++}^{L}$. (we assume this throughout).

Hicksian demand and expenditure function

- The solution to EMP, denoted by $h(p, u)$, is called the Hicksian or compensated demand function
- The value of EMP, denoted by $e(p, u)$, is called the expenditure function
- The following observation is crucial:

Proposition

Fix a price vector $p \in \mathbb{R}^L_{++}$ and a continuous utility function $u(x)$ representing locally non-satiated preference relation.
i) If $x^* = x(p, w)$, then $x^* = h(p, u(x^*)) = h(p, v(p, w))$.
ii) If $x^* = h(p, u)$, then $x^* = x(p, p \cdot x^*) = x(p, e(p, u))$.

Summarizing:

$$x(p,w) = h(p, v(p,w)) \text{ and } h(p,u) = x(p, e(p,u)).$$

Obviously then also:

$$w = e(p, v(p,w)) \text{ and } u = v(p, e(p,u)).$$

Properties of the expenditure function

It is straightforward to derive the following properties:

Theorem

Suppose $u(\cdot)$ *is a continuous representation of a locally nonsatiated preference relation defined on* $\mathbb{R}^L_+$*. The expenditure function* $e(p,u)$ *is*

- *Homogeneous of degree one in* p
- *Stirctly increasing in* u *and nondecreasing in* p_l *for any* l
- *Concave in* p
- *Continuous in* p *and* u

Properties of the Hicksian demand function

Theorem

Suppose $u(\cdot)$ *is a continuous representation of a locally nonsatiated preference relation defined on* $\mathbb{R}^L_+$*. Then for any* $p \gg 0$*, the Hicksian demand correspondence* $h(p,u)$ *possesses the following properties:*

- *Homogeneity of degree zero in* p
- *No excess utility: For any* $x \in h(p,u)$, $u(x) = u$
- *Convexity/uniqueness: If preferences are convex, then* $h(p,u)$ *is a convex set. If preferences are strictly convex, then* $h(p,u)$ *is a singleton.*

Deriving Hicksian demand from expenditure function

- Given Hicksian demand $h(p, u)$, the expenditure function is simply $e(p, u) = p \cdot h(p, u)$
- In the other direction, we have:

Theorem

Suppose $u(\cdot)$ is a continuous representation of a locally nonsatiated and strictly convex preference relation defined on $\mathbb{R}^L_+$. For all p and u, we have:

$$h(p, u) = \nabla_p e(p, u) .$$

Proof similar to Roy's identity.

Price derivatives of the Hicksian demand

- Denote by $D_p h(p, u)$ the $L \times L$ matrix of derivatives of $h(p, u)$ w.r.t. prices
- Given continuous, nonsatiated, and strictly convex preferences represented by a continuous $u(x)$, the derivative matrix $D_p h(p, u)$ is negative semidefinite, symmetrix, and satisfies $D_p h(p, u)\, p = 0$.
- Negative semidefiniteness is the compensated law of demand:

$$dp \cdot D_p h(p, u)\, dp \leq 0 \text{ for all } dp, \;\text{ that is } dp \cdot dh \leq 0.$$

- Note that the compensation is different from the previous lecture:
  - Hicksian demand gives optimal consumption when wealth is adjusted in order to keep utility level constant
  - In the previous lecture, we had a different compensation: change wealth in order to maintain original consumption bundle just affordable

Slutsky Equation

- Hicksian demand is not directly observable, whereas Walrasian demand in principle is
- Nevertheless, price derivative of Hicksian demand can be derived from Walrasian demand:

Theorem

Suppose $u(\cdot)$ is a continuous representation of a locally nonsatiated and strictly convex preference relation defined on $\mathbb{R}^L_+$. Then, for all (p, w) and $u = v(p, w)$, we have

$$D_p h(p, u) = D_p x(p, w) + D_w x(p, w) \, x(p, w)^T$$

Some comments

- Slutsky equation translates properties of $D_p h(p, u)$ into restrictions on Walrasian demand (in particular, the negative semidefiniteness of $D_p h(p, u)$)
- Recall the properties of $S(p, w)$, the substitution matrix defined in the choice-based approach
- $D_p h(p, u)$ is symmetric but $S(p, w)$ need not be.
- The restrictions under the preference-based approach are stronger, so we obtain additional observable implications.

Integrability

- utility maximization implies: (1) Walras' Law, (2) Homogeineity of degree 0 for the demands and, (3) $S(p, w)$ is symmetric and negative semidefinite (s.n.s.d.).
- can we also conclude that (1)-(3) imply that there exists a rational preference relation that could have generated $x(p, w)$
- the integrability problem: the answer is yes
- (1)-(3) are the only consequences of preference maximization
- The route we follow: $x(p, w) \longrightarrow e(p, u) \longrightarrow \succeq$

- consider first $x(p, w) \longrightarrow e(p, u)$
- recall $x(p, w) = x(p, e(p, u)) = h(p, u)$, so

$$\nabla_p e(p, u) = x(p, e(p, u))$$

Note:

- a system of PDE with $e(p^0, u) = w^0$ as an initial condition. Solved for e as a function of p along a given indifference surface
- if there exists a solution $e(p) = e(p, u)$, then we must have

$$D_p^2 e(p) = S(p, e(p))$$

- where $S(p, e(p))$ is symmetric, negative semidefinite (s.n.s.d.) Slutsky evaluated along the solution
- s.n.s.d. Slutsky is also sufficient condition by the Frobenius' Theorem

- consider then $e(p,u) \longrightarrow \succeq$
- Recall the properties of $e(p,u)$:
    - i) $e(p,u)$ is homogenous of degree 1 in p.
    - ii) Strictly increasing in u and non-decreasing in p_l for all l.
    - iii) Concave in p.
    - iv) Continuous in p, u.
- Again we have conversely:

Proposition

Suppose that $e(p,u)$ satisfies i)-iv). Then there exists a utility function $u : \mathbb{R}^L_+ \rightarrow \mathbb{R}$ such that $e(p,u)$ is the expenditure function of the associated EMP. Furthermore, the upper level sets for this utility function are given by

$$V_u = \left\{ x \in \mathbb{R}^L_+ \,\middle|\, p \cdot x \geq e(p,u) \text{ for all } p \in \mathbb{R}^L_{++} \right\}.$$

Then also, $e(p,u) = \min\{p \cdot x \,| x \in V_u\}$.

where we have set $w = e(p,u) = e(p, v(p,w))$.

Theorem

(Integrability)
Suppose that $x\left(p, w\right)$ is homogenous of degree 0, satisfies Walras' law, and has a symmetric and negative semi-definite substitution matrix. Then there is a (quasiconcave) utility function $u\left(x\right)$ such that $x\left(p, w\right)$ is the solution to UMP.

Welfare evaluation of a price change

- How to quantify the welfare effect of a change in the economic environment?
- For simplicity, consider a change in the prices
- Remember that utility is an ordinal concept, so it is not possible to give an unambiguous measure of welfare
- However, if we know preferences, we can use any indirect utility function to rank different price-wealth situations
- Moreoever, we can choose some indirect utility function that measures welfare changes in monetary units
- These are called money metric indirect utility functions

Money metric indirect utility function

- Start with any indirect utility function $v(p, w)$
- Choose any price vector $\overline{p} \gg 0$
- Then, function $e(\overline{p}, v(p, w))$ is also an indirect utility function
- It gives the wealth required to reach utility $v(p, w)$ at prices $\overline{p}$
- Thus, $e\left(\overline{p}, v\left(p^1, w\right)\right) - e\left(\overline{p}, v\left(p^0, w\right)\right)$ is a measure of welfare due to price change from p^0 to p^1 expressed in monetary units
- Note that the particular choice of v does not make any difference - only the properties of the preferences and the choice of $\overline{p}$ matter

- How to choose $\overline{p}$?
- Two natural possibilities: $\overline{p} = p^0$ and $\overline{p} = p^1$
- The former is the equivalent variation (EV), the latter the compensated variation (CV)
- Formally, writing $u^0 = v\left(p^0, w\right)$ and $u^1 = v\left(p^1, w\right)$, we have:

$$\begin{aligned} EV\left(p^0, p^1, w\right) &= e\left(p^0, u^1\right) - e\left(p^0, u^0\right) = e\left(p^0, u^1\right) - w \\ CV\left(p^0, p^1, w\right) &= e\left(p^1, u^1\right) - e\left(p^1, u^0\right) = w - e\left(p^1, u^0\right) \end{aligned}$$

- We have: $v\left(p^0, w + EV\right) = u^1$ and $v\left(p^1, w - CV\right) = u^0$

- We can also write EV and CV using Hicksian demand curve
- Suppose that only the price of good 1 changes ($p_l^0 = p_l^1 = \overline{p}_l$ for all $l \neq 1$). Then:

$$\begin{aligned} EV\left(p^0, p^1, w\right) &= \int_{p_1^1}^{p_1^0} h_1\left(p_1, \overline{p}_{-1}, u^1\right) dp_1 \\ CV\left(p^0, p^1, w\right) &= \int_{p_1^1}^{p_1^0} h_1\left(p_1, \overline{p}_{-1}, u^0\right) dp_1 \end{aligned}$$

- We have $EV\left(p^0, p^1, w\right) > (<) \; CV\left(p^0, p^1, w\right)$ if good 1 is normal (inferior)
- What kind of preferences give $EV\left(p^0, p^1, w\right) = CV\left(p^0, p^1, w\right)$?