Microeconomic Theory I

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Lecture 2: Choice-based approach to consumer theory

- In this lecture we start with some natural assumptions on choice behavior and ask what implications they have on consumption
- In particular, we analyze the properties of Walrasian demand function
- Exogenous variables: prices and income
- Endogenous variables: consumption choices
- Questions:
  - How do endogenous variables change in response to exogenous variables?
  - How complete is the description of the consumer behavior that can be obtained by applying WA?
- Main material for this lecture: MWG Ch. 2 (and parts of Ch. 3)

Framework:

- X the choice set, the consumption set
- We take $X = \mathbb{R}^L_+$, where $L \in \mathbb{N}$.
- Means that goods are divisible
- A commodity vector is a list of consumption of each commodity: $x = (x_1, ..., x_L)$, where each $x_l \in \mathbb{R}_+$ for each $l \in \{1, ..., L\}$.

Budget set

- All commodities are traded at given prices p (complete markets)
- The feasible budget is defined by p and income or wealth w.
- A budget feasible consumption is one that can be purchased with the disposable income.
- In classical consumer theory (and in this lecture), we assume that prices are linear:
  - The price of an additional unit of good l is independent of the amount of good l purchased.
  - The price of an additional unit of good k is independent of consumptions of goods $l \neq k$.
  - Rules out quantity discounts, bundling etc.
- $p \in \mathbb{R}^L_+,\ w \in \mathbb{R}_+$.

Walrasian budget set

- We define the Walrasian budget set:

$$B = \{x \in X \,|\, p \cdot x \leq w\},$$

  where $p \cdot x = \Sigma_{l=1}^{L} p_l x_l$
- Since B is determined by p and w, we write it as $B(p, w)$.
- $B(p, w)$ rules out nonlinearities, indivisibilities, uncertainties, and interdepencies between individuals
- $B(p, w)$ is convex

The choice rule

- Following the previous lecture, we can write the choice structure as $\left(\mathcal{B}^W, C\left(\cdot\right)\right)$, where

$$\mathcal{B}^W \equiv \left\{B\left(p, w\right) : p \in \mathbb{R}_+^L,\ w \in \mathbb{R}_+\right\}$$

  is the family of Walrasian budget sets.
- Note that $\mathcal{B}^W$ contains only a special class of subsets of X. It certainly does not include all two- and three element subsets of X. (why is this fact relevant?)
- We denote the choice rule by $x\left(p, w\right) \equiv C\left(B\left(p, w\right)\right)$. This is called the *walrasian demand correspondence*.
- When $x\left(p, w\right)$ is single valued (which we take as given in this lecture), we call it the demand function $x : \mathbb{R}_+^{L+1} \rightarrow \mathbb{R}_+^L$.
- We make two important assumptions regarding $x\left(p, w\right)$:

Walras' law

Assumption 1: the Walras' Law:

$$p \cdot x\left(p, w\right) = w \text{ for all } p \gg 0, w > 0.$$

- All income is used
- Sometimes called the adding-up restriction

Homogeneity of demand:

Assumption 2: Homogeneity restriction

$$x(\lambda p, \lambda w) = x(p, w) \text{ for all } \lambda > 0 \text{ and all } p, w.$$

- that is, $x(p, w)$ is homegenous of degree zero in (p, w)
- Intuitive, because doubling prices and income will not change budget set
- No money illusion

Comparative statics

Wealth effects

- Denote by $D_w x(p, w) \in \mathbb{R}^L$ the derivative of $x(p, w)$ with respect to w
- This is a vector containing the wealth effects of demand
- A commodity l is normal at (p, w) if $\partial x_l (p, w) / \partial w \geq 0$, otherwise l is inferior at (p, w)

Price effects

- Denote by $D_p x (p, w) \in \mathbb{R}^{L \cdot L}$ the derivative of $x (p, w)$ with respect to prices
- This is a $L \times L$ matrix: ijth element is effect of price change of good j on the demand of good i
- Good l is a Giffen good at (p, w) if $\partial x_l (p, w) / \partial p_l > 0$

Implications of the Walras' law

Make sure you understand the meaning of the following expressions (it is good excercise to work these out in detail):

- Engel aggregation

$$p \cdot D_w x(p, w) = 1. \tag{1}$$

- Cournot aggregation

$$p \cdot D_p x(p, w) + x\,(p, w)^T = 0 \tag{2}$$

Implications of the homogeneity restriction:

$$D_p x\left(p, w\right) p + D_w x\left(p, w\right) w = 0. \tag{3}$$

Implications for elasticities

- Denote the budget share of commodity l by

$$b_l(p,w) = p_l x_l(p,w)/w$$

- Denote price elasticities by

$$\varepsilon_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k}\frac{p_k}{x_l(p,w)}$$

- Denote income elasticity by

$$\varepsilon_{lw}(p,w) = \frac{\partial x_l(p,w)}{\partial w}\frac{w}{x_l(p,w)}$$

- Engel and Cournot aggregation imply, respectively:

$$\Sigma_{l=1}^{L} b_l\left(p, w\right) \varepsilon_{lw}\left(p, w\right) = 1$$

$$\Sigma_{l=1}^{L} b_l\left(p, w\right) \varepsilon_{lk}\left(p, w\right) + b_k\left(p, w\right) = 0$$

- Homogeneity restriction implies

$$\Sigma_{k=1}^{L} \varepsilon_{lk}\left(p, w\right) + \varepsilon_{lw}\left(p, w\right) = 0$$

Weak axiom

- What is implied by the WA in conjuction with the Walras law and homogeneity restriction?
- Recall the definition of WA from the first lecture:

Definition

$(\mathcal{B}, C(\cdot))$ satisfies the weak axiom of revealed preference (WA) if the following property holds:
If $x, y \in B$ and $x \in C(B)$, then for all B' such that $x, y \in B'$ and $y \in C(B')$, we have $x \in C(B')$.

- In the context of Walrasian budget sets WA takes the form:

Definition

$x(p, w)$ satisfies WA if for any (p, w) and (p', w'), the following holds:

$$\text{If } p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w), \text{ then } p' \cdot x(p, w) > w'.$$

- Can you see that the two definitions are the same?

Implications of WA for Walrasian demand

- Law of Demand means: $x_l(p, w)$ is decreasing in p_l.
- But this does not necessarily hold (Giffen goods)
- Why not? An increase in p_l changes relative prices (slope of the budget line) and effective wealth (i.e. is not feasible with new prices).
- To isolate the substitution effect we consider compensated price changes
- Then we obtain the *compensated* law of demand
- Idea: Look at the effects of relative price changes by forcing the original consumption point to lie on the new budget line.
- Formally (p', w') is a compensated price change if $p' \cdot x(p, w) = w'$
- Wealth adjustment corresponding to price change Δp is $\Delta w = \Delta p \cdot x(p, w)$
- This is called Slutsky wealth compensation

WA and compensated law of demand

Theorem

Suppose $x(p, w)$ satisfies Assumptions 1-2 (Walras' law and homogeneity). Then, for compensated price changes (p', w'), the following statements are equivalent:

1. $x(p, w)$ *satisfies WA*
2. $(p' - p)(x(p', w') - x(p, w)) \leq 0$, *where the inequality is strict whenever* $x(p, w) \neq x(p', w')$.

Proof: In MWG and discussed in class.

- The compensated law of demand, $\Delta p \cdot \Delta x \leq 0$, has implications for the substitution matrix or the Slutsky matrix of $x(p, w)$.
- The differential analog $dp \cdot dx \leq 0$ implies

$$dp \cdot [D_p x(p, w) + D_w x(p, w) x(p, w)^T] dp \leq 0.$$

- This says that $L \times L$ matrix in brackets is negative semidefinite. Denote the Slutsky matrix by $S(p, w)$.

Theorem

Suppose $x(p, w)$ is differentiable in p and w, and satisfies Assumptions 1-2 and the WA. Then, at any (p, w), the Slutsky matrix $S(p, w)$ is negative semidefinite.

Proof: this follows in a straightforward manner from the previous result.

Some comments:

- Substitution effect w.r.t. change in own price unambiguous
- Giffen goods must be inferior
- $S(p, w)$ is not symmetric in general
- has rank less than L

Preliminaries for the Preference-based approach to consumer demand

From lecture 1:

- If $\succeq$ is rational, then $C^*(\mathcal{B}, \succeq)$ satisfies WA.
- Let $x(p, w) = C^*(B(p, w), \succeq)$. We call $x(p, w)$ the Walrasian demand of the consumer with preferences $\succeq$.
- If $x(p, w)$ is Homogenous of degree 0 in (p, w) and satisfies Walras' Law, then $x(p, w)$ also satisfies compensated law of demand and hence the Slutsky matrix is negative semidefinite.
- When will Homogeneity of degree 0 and Walras' Law be satisfied?

Homogeneity of Degree 0:
This follows from the fact that for all $\lambda > 0$,

$$B(p, w) = B(\lambda p, \lambda w)$$

Walras' Law:

We need new assumptions on consumer's tastes:

Definition

Preference relation $\succsim$ is locally nonsatiated if $\forall x \in X$, and $\forall \delta > 0$, there is $y \in X$ such that $\|y - x\| < \delta$ and $y \succ x$.

- $x(p, w)$ satisfies Walras' Law if $\succeq$ is locally non-satiated (why?)
- What is the relationship between locally nonsatiated and monotonic preferences?

- Finally, what would guarantee the existence of a well defined demand *function*? That is, when is $x(p, w)$ single valued?
- Key: convexity assumptions on $\succeq$.

Definition

$\succeq$ is said to be

i) Convex if for all $x, y, \in X$ and for all $t \in [0, 1]$, we have

$$x \succeq y \Rightarrow (tx + (1-t)y) \succeq y$$

ii) Strictly Convex if for all $x, y, \in X$ and for all $t \in (0, 1)$, we have

$$x \succeq y \Rightarrow (tx + (1-t)y) \succ y$$

Assume next that a utility function u representing $\succeq$ exists. Recall the definition of quasi-concave functions:

Definition

Let f be defined on the convex set $X \subset \mathbb{R}^n$. It is a quasiconcave function if and only if

$$f(tx_1 + (1-t)x_2) \geq \min[f(x_1), f(x_2)]$$

for every $x_1, x_2 \in X$,and $0 \leq t \leq 1$.

It is a straightforward exercise to show the following alternative characterization for quasiconcave functions:

Theorem

Let f be defined on the convex set $X \subset \mathbb{R}^n$. It is said to be **quasiconcave** *if its upper-level sets*

$$U(f, \alpha) = \{x : x \in X, f(x) \geq \alpha\}$$

are convex sets for every real α.

- Notice from here the connection between quasiconcavity of a representation and the convexity of the underlying preferences.
- It is clear that a concave function is also quasiconcave, since

$$\begin{aligned} f(tx_1 + (1-t)x_2) &\geq \\ tf(x_1) + (1-t)f(x_2) &\geq \min\left[f(x_1), f(x_2)\right]. \end{aligned}$$

Note, quasiconcavity (unlike concavity) is an "ordinal" property. Show the following:

Theorem

If $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconcave and if $g : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $g(f(x))$ is also quasiconcave.

These definitions can be extended to the strict case in a straightforward manner:

Definition

Let f be defined on the convex set $X \subset \mathbb{R}^n$. It is said to be strictly quasiconcave if

$$f(tx_1 + (1-t)x_2) > \min[f(x_1), f(x_2)]$$

for every $x_1, x_2 \in X, x_1 \neq x_2$, and $0 < t < 1$.

Strict quasiconcavity of a representation is equivalent to strict convexity of preferences.

Proposition

i) If $\succeq$ is convex, then $x(p, w)$ is a convex set for all p, w. ii) If $\succeq$ is strictly convex, then $x(p, w)$ is a singleton for all p, w.

Proof: this follows from definition of $C^*(\mathcal{B}, \succeq)$, using convexity of the budget set and $\succeq$.