Microeconomic Theory I

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Lecture 1: Preference, choice, and utility representation

This lecture introduces the basic concepts of decision theory in an abstract setting. We will cover:

- Basic elements of choice theory
- Properties of preference relations
- Utility representation
- Material for this lecture:
  MWG Ch. 1, Ch. 3.C
  Kreps Ch. 1-3

Two approaches to individual choice behavior:

1. Take the *choice behavior* as the primitive characteristic of the decicion maker
2. Take the *tastes* as the primitive characteristic

- The former starts by making assumption directly on choice behavior, while latter starts by formalizing the preference relation
- In this lecture we will introduce both of these approaches in an abstract setting, and consider their relations with each other
- In the succeeding lectures we use these to develop the consumer theory

Choice Rules

- Let us start by describing the alternatives from which the choice is to be made:
- Let X denote the set of all alternatives
- Let $B \subset X$ denote the feasible set
- Here X is the "universe" of all alternative choices, while B is the set of alternatives available in a given situation.
- This distinction is important, because we want to model choice behavior across different situations. An important example is:
  $X = \mathbb{R}_+^n$ is the set of possible consumption bundles.
  $B = B(p, w) = \{x \in \mathbb{R}_+^n : p \cdot x \leq w\}$ is the budget set.

Choice structure

Choice structure $(\mathcal{B}, C(\cdot))$ consists of two elements:

1. $\mathcal{B}$ is the collection of all possible feasible sets
   - A family of non-empty sbsets of X
   - Lists all possible choice situations, for which the choice behavior is to be defined
   - $\mathcal{B}$ does not necessarily contain all subsets of X
2. $C(B)$ is the choice correspondence such that $C(B) \subset B$ for all $B \in \mathcal{B}$
   - Contains the set of acceptable alternatives from B
   - Must be non-empty
   - May contain many elements

Weak axiom of revealed preference

- We can define all kinds of choice behavior by appropriately choosing $\mathcal{B}, C(\cdot)$
- E.g. we could have $C(\{x, y\}) = x$ and $C(\{x, y, z\}) = y$
- What should we "assume", if we somehow want to restrict to "consistent" behavior?
- The idea of requiring revealed preferences to hold is roughly: if x is weakly preferred over y in a given situation, then this preference should continue to hold in other situations where both are available. Formally:

Definition

$(\mathcal{B}, C(\cdot))$ satisfies the weak axiom of revealed preference (WA) if the following property holds: If $x, y \in B$ and $x \in C(B)$, then for all B' such that $x, y \in B'$ and $y \in C(B')$, we have $x \in C(B')$.

From choice to preferences

- For a given $(\mathcal{B}, C(\cdot))$, we can define the revealed preference relation $\succeq^*$by

$$x \succeq^* y \iff \text{there is some } B \in \mathcal{B} \text{ with } (x,y) \in B \text{ and } x \in C(B).$$

- Note that $\succeq^*$ is not necessarily defined for all pairs x, y.
- We already see that choice behavior leads to something that we can call preferences
- To be more precise, let us go back and start from a different perspective: take preferences as our primitive

Preference Relations

Binary relations

- Technically, a preference relation is a binary relation
- A binary relation R is defined by writing $R \subseteq X \times X$, where $(x,y) \in R$ if the ordered pair (x,y) is in the relation R
- That is to say, a binary relation is simply a subset of $X \times X$
- Write xRy to mean $(x,y) \in R$ and $x\tilde{R}y$ to mean $(x,y) \notin R$

Some properties of binary relations:

- A given binary relation may or may not have various properties, e.g., R is
  - reflexive if xRx for all $x \in X$
  - irreflexibe if $x\tilde{R}x$ for all $x \in X$
  - symmetric if xRy implies yRx
  - asymmetric if xRy implies $y\tilde{R}x$
  - antisymmetric if xRy and yRx imply $x = y$
  - transitive if xRy and yRz imply xRz
  - negatively transitive if $x\tilde{R}y$ and $y\tilde{R}z$ imply $x\tilde{R}z$
  - complete if either xRy or yRx for all $x, y \in X$

- So, let us take a binary relation, and denote it by $\succsim$
- Interpretation: $x \succsim y$ means "x is at least as good as y"
- Think of $\succsim$ as the primitive, and use it to define two other relations, $\succ$ and $\sim$:
  - $x \succ y \iff (x \succsim y \text{ and } y \not\succsim x)$
  - $x \sim y \iff (x \succsim y \text{ and } y \succsim x)$
- Prefrences are *rational* if they satisfy two axioms:

Definition

A binary relation $\succsim$ is a rational preference relation, if it is 1) complete, and 2) transitive

Some other properties of rational preferences:

- If $\succsim$ is complete and transitive, then
  - $\succsim$ is reflexive
  - $\succ$ is irreflexive, transitive, asymmetric, negatively transitive
  - $\sim$ is reflexive, transitive, symmetric
  - if $x \succ y \succsim z$, then $x \succ z$
- It is a good exercise to prove these
- Note that we could just as well have started with $\succ$ instead of $\succsim$ (what should we then require of $\succ$?)

From preferences to choice

- Given a preference relation, define:

$$c^*(B, \succeq) = \{x \in B : x \succeq y, \text{ for all } y \in B\}.$$

- This defines the decision-maker's most preferred alternatives in B
- Assuming that $c^*(B, \succeq)$ is non-empty for all $B \in \mathcal{B}$, then $(\mathcal{B}, c^*(\cdot, \succeq))$ is a choice structure
- We say that $\succeq$ generates the choice structure $(\mathcal{B}, c^*(\cdot, \succeq))$

Connections between the choice- and preference approaches

- What exactly is the relationship between the two approaches?
- When is a given choice behavior consistent with rational preferences, i.e. can be rationalized?
- We have seen that choice structure leads to $\succeq^*$
- But $\succeq^*$ does not need to be complete or transitive
- What if the choice structure satisfies WA?
- On the other hand, $\succeq$ leads to $c^*(B, \succeq)$
- If $\succeq$ is rational, does $c^*(B, \succeq)$ satisfy WA?
- It turns out that WA and rational preferences are almost equivalent

- Rational preferences imply WA:

Theorem

If $\succeq$ is a rational preference relation, then the choice structure $(c^*(\cdot, \succeq), \mathcal{B})$ induced by $\succeq$ satisfies the weak axiom.

- Proof is straightforward.

- In the other direction, we need a qualification:

Theorem

Let $\mathcal{B}$ include all subsets of X with two and three elements. Then, if $(\mathcal{B}, C(\cdot))$ satisfies WA, the induced revealed preference relation $\succeq^*$ is rational. Moreover, this is the only preference relation that induces $(\mathcal{B}, C(\cdot))$.

- Key in the proof: choice must be defined for all three-element subsets in order to establish transitivity of the revealed preference relation.

Why is the restriction on the sets in the second theorem important? Check the following examples:

- Ex.1:
  $X = \{x, y, z\}$, $\mathcal{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$
  $C(\{x, y\}) = x, C(\{y, z\}) = y, C(\{x, z\}) = z$
- Ex.2: As Ex. 1 but add X to $\mathcal{B}$

Some comments:

- It can be argued that real life decicison making is more complicated than assumed here
- Can we expect rationality to hold in reality? Problems:
  - Framing; Joint decisions and aggregation across individuals; Decision making is costly
- Literature on "bounded rationality" investigates the implications of various contraints in the actual decision making processes (sequential procedures, limited memory, choice of information, etc.). See e.g. Ariel Rubinstein: "Modeling Bounded Rationality" at http://arielrubinstein.tau.ac.il/books.html)
- Typically such procedures violate rationality as defined here, but on the other hand some "heuristic" choice procedures can be "rationalized" (e.g. Simon's satisficing procedure - see excercise set)
- An example of new work that extends the scope of choice theory is Bernheim and Rangel: "Beyond revealed preference: choice theoretic foundations for behavioral welfare economics" (http://www.nber.org/papers/w13737)

Utility Representation

- In most models we work with a utility function for convenience: it can be manipulated using standard math tools
- Is it OK to let a real-valued function to represent potentially complicated preferences over the choice set?
- What are we exactly assuming when taking this approach?
- The objective: analyze the relationship between the axioms and the utility function
- **Representation for $\succeq$**
  We are looking for numerical representation of $\succeq$, which is a function $u : X \to \mathbb{R}$ such that

$$u(x) \geq u(y) \iff x \succeq y. \tag{1}$$

Theorem

If there exists utility function representing $\succeq$, then $\succeq$ is rational.

- Proof: u maps into real line on which the binary relation "$\geq$" is complete and transtive.

Theorem

If the choice set X is finite and $\succeq$ is rational, then $\succeq$ has a representation.

- Constructive proof via induction (discussed in the class)
- Note that if u represents $\succeq$, then so does $f \circ u$ for any increasing $f : \mathbb{R} \to \mathbb{R}$

- The finiteness restriction on X that guarnatees existence of representation is often too demanding (e.g. consumption space)
- Example: Lexicographic preferences. $X = [0,1] \times [0,1]$. Let

$$(x_1, x_2) \succeq (y_1, y_2)$$
$$\Longleftrightarrow$$
$$x_1 > y_1 \text{ or } [x_1 = y_1 \text{ and } x_2 \geq y_2].$$

Assuming a representation u for these preferences leads to a contraction. We need further restrictions on the preference relation

Continuous preferences

- Let $X \subset \mathbb{R}^L_+$.

Definition

The preference relation $\succeq$ on X is continuous if it is preserved under limits: for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succeq y^n$ for all n, $x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$, we have $x \succeq y$.

- There is another equivalent definition: The preference relation $\succeq$ is continuous if for all $x \in X$, the sets $\{y \in X : y \succeq x\}$ and $\{y \in X : x \succeq y\}$ are closed.
- Can you show that these two definitions are equivalent?
- Are lexicographic preferences continuous?

Theorem

Let $X \subset \mathbb{R}^L_+$. If $\succeq$ defined on X is rational (that is, complete and transitive) and continuous, then there exists a continuous utility function $u(x)$ that represents $\succeq$.

- Proof is technical. A somewhat simplified version is proven in MWG.

Some basic properties of preferences and implications for utility representation

Definition

Preference relation $\succeq$ is monotone if $x \in X \subset \mathbb{R}^L_+$ and $y \gg x$ implies $y \succ x$. It is strongly monotone if $y \geq x$ and $y \neq x$ imply that $y \succ x$.

- Monotone preferences imply increasing $u(x)$

Definition

Preference relation $\succeq$ is convex if for every $x \in X$, the set $\{y \in X; y \succeq x\}$ is convex.

Definition

Preference relation $\succeq$ is strictly convex if for every $x \in X$, we have $y \succeq x$, $z \succeq x$, and $y \neq z$, imply $\alpha y + (1-\alpha) z \succ x$ for all $\alpha \in (0,1)$.

- (strictly) convex preferences imply (strictly) quasiconcave $u(x)$

Definition

A monotone preference relation $\succeq$ is homothetic if $x \sim y$ implies $\alpha x \sim \alpha y$ for any $\alpha > 0$.

- There is a utility representation that is homogenous of degree one.

Definition

Preference relation $\succeq$ on $X = (-\infty, \infty) \times \mathbb{R}^{L-1}_+$ is quasilinear with respect to commodity 1 if
$x \sim y$ implies $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, ..., 0)$ and any $\alpha > 0$.
$x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

- There is a utility representation of the form $u(x) = x_1 + \phi(x_2, ..., x_L)$.