FDPE Macroeconomics 2009, lecture notes 5

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1 Stationary equilibria in incomplete market models

Lecture notes 4 presented examples of individual problems that are stochastic because agents cannot insure themselves by trading with assets that are contingent on idiosyncratic shocks. In this lecture note, we show how such individual problems can be extended to general equilibrium models. The key concept is that of a stationary equilibrium (or *recursive* stationary equilibrium). In the stationary equilibrium, aggregate variables are constant over time. Yet at the same time there may be a lot of movement going on at the individual level as individuals are hit by idiosyncratic shocks. We also revisit the question of the optimal capital tax rate, explain how to find a stationary equilibrium in practice, and give some references for the related literature.

1.1 The infinite horizon savings model with production

The individual problem (see lecture notes 4, section 1.2) reads as

$$V(a,s) = \max_{a'}\{u(c) + \beta E V(a',s')\} \tag{1}$$

subject to

$$\begin{aligned} c + a' &= (1+r)a + ws \quad (2) \\ a' &\geq \underline{a} \quad (3) \end{aligned}$$

where a is a financial asset and $s \in S = \{s_1, s_2, ..., s_{n_s}\}$ is an income shock following a first-order Markov chain with $P(s', s)$ denoting the transition probability matrix. Let $g(a, s)$ denote the associated optimal savings policy (optimal a' given state). Similarly, let $c(a, s)$ denote the optimal consumption policy.

One way to make the interest rate and wage rate endogenous is to extend the model with production and an aggregate resource constraint (Aiyagari, 1994). One may also consider a pure exchange economy (Hugget, 1993). Here we consider the first possibility. So let us assume that there is a representative firm that combines capital and labor to produce output goods. The constant-returns-to-scale technology is given by $f(k, n)$ where k and n are the aggregate stock of capital and aggregate labor supply. In this case, the labor supply is exogenous and fixed, so we can normalize it to one. The aggregate capital stock in turn is determined endogenously as households' aggregate holdings of the financial asset. The standard firm problem then implies that the interest rate and the wage rate are given by the marginal productivity of capital less the depreciation rate of capital and the marginal productivity of labor, respectively.

In order to formally define the stationary equilibrium, let us first assume (following LS, chapter 17), that the individual is constrained to choose asset holdings from a discrete set $A^d = \{a_1, a_2, ..., a_{n_a}\}$. We also assume that there is a continuum of households with total mass equal to one.

Let λ_t denote the distribution of households over state variables in period t. Then $\lambda_t(a, s)$ gives the unconditional probability that a randomly picked household in period t has assets $a \in A^d$ and income shock $s \in S$. In other words, $\lambda_t(a, s)$ gives the mass of households in state (a, s).

Given the distribution λ_t and the policy function $g(a, s)$, the mass of households in state (a', s') (where $a' \in A^d$ and $s' \in S$) in period $t + 1$ can be computed as follows:

$$\lambda_{t+1}(a', s') = \sum_j \sum_i \lambda_t(a_i, s_j) P(s', s_j) I(a', a_i, s_j) \tag{4}$$

where $I(a', a_i, s_j) = 1$ if $g(a_i, s_j) = a'$ and 0 otherwise. This may be written (with some abuse of notation) as

$$\lambda_{t+1}(a', s') = \sum_{s} \sum_{a:a'=g(a,s)} \lambda_t(a, s) P(s', s). \tag{5}$$

The time-invariant distribution λ that solves equation (5) for all a' and s', is the stationary distribution we are looking for.

We can now define the stationary equilibrium. The stationary equilibrium consists of a time invariant distribution $\lambda(a, s)$, capital stock k and aggregate consumption c, prices r and w, policy functions $g(a, s)$ and $c(a, s)$ such that:

i) The policy functions solve the household problem.

ii) Prices satisfy

$$r = f_k(k, 1) - \delta \tag{6}$$

$$w = f_n(k, 1) \tag{7}$$

iii) Aggregate capital stock and consumption are given by average household asset holdings and consumption:

$$k = \sum_{s} \sum_{a} \lambda(a, s) g(a, s) \tag{8}$$

$$c = \sum_{s} \sum_{a} \lambda(a, s) c(a, s) \tag{9}$$

iv) Aggregate resource constraint is satisfied:

$$c + \delta k = f(k, 1) \tag{10}$$

v) The probability distribution λ is time-invariant (i.e. stationary):

$$\lambda(a', s') = \sum_{s} \sum_{a:a'=g(a,s)} \lambda(a, s) P(s', s) \tag{11}$$

for all s' and a'.

When we allow for a continuous savings decision, we need to integrate over households' asset holdings, instead of summing up over a set of discrete asset levels. This makes the formal definition of the stationary equilibrium somewhat more complicated.

It is now convenient to define $X = [\underline{a},\overline{a}] \times S$, where $\overline{a}$ is the upper bound on the asset holdings that we impose when solving the individual problem (it should never be binding). We denote a typical element of X by x. The distribution is now given by $\lambda(x)$ and optimal policies by $g(x)$ and $c(x)$. We define a probability transition measure Q for pairs x, B, where $B = \{s'\} \times A \subset [\underline{a},\overline{a}]$ by saying that $Q(x, B)$ equals $P(s', s)$ if $g(x) \in A$ and zero otherwise. That is, $Q(x, B)$ gives the probability that an individual in state x moves to a state that belongs to the set B.

Condition iii) and v) above can now be rewritten as:

$$k = \int g(x)\lambda(x)dx \tag{12}$$

$$c = \int c(x)\lambda(x)dx \tag{13}$$

and

$$\lambda(B) = \int Q(x, B)\lambda(x)dx \tag{14}$$

for all B.

1.2 A stochastic OLG model with production

The life cycle savings model presented in section 1.3 of the previous lecture notes is also easy to extend to a general equilibrium overlapping generations economy using the concept of stationary equilibrium.

We write the individual problem recursively as follows:

$$V(a, s, j) = \max_{a'}\{u(c) + \beta EV(a', s', j + 1)\} \tag{15}$$

subject to

$$c + a' = (1 + r)a + ws \tag{16}$$

$$a' \geq 0. \tag{17}$$

where $j \in \{1, 2, ..., J\}$ refers to individual's age. Let $g(x, j)$ denote the optimal policy given assets, income shock, and age and let $\lambda(x, j)$ be the distribution function. Define

$X = [0,\overline{a}] \times S$. Again, a typical element of X is denoted by x. As above, define a probability transition measure Q for pairs x, B where $B = \{s'\} \times A \subset [0,\overline{a}]$. We define $Q(x,B,j)$ by saying that it equals $P(s',s)$ if $g(x,j) \in A$ and zero otherwise. We normalize the total population to one. In order to replace those who die, we must assume that every period, a mass of $1/J$ of new individuals are born so that the population remains constant over time. Let $\pi(s)$ denote the share of new born individuals with income shock s.

The stationary equilibrium consists of a distribution λ, value function v and associated optimal policy functions $g(x,j)$ and $c(x,j)$, and prices r and w such that:

i) The value and policy functions solve the household problem.

ii) Prices satisfy

$$\begin{aligned} r &= f_k(k,1) - \delta \qquad (18) \\ w &= f_n(k,1) \qquad (19) \end{aligned}$$

iii) Aggregate capital stock and consumption are given by average household asset holdings and consumption:

$$\begin{aligned} k &= \int g(x,j)\lambda(x,j)dxdj \qquad (20) \\ c &= \int c(x,j)\lambda(x,j)dxdj \qquad (21) \end{aligned}$$

iv) Aggregate resource constraint is satisfied:

$$c + \delta k = f(k,1) \qquad (22)$$

v) The probability distribution λ is time-invariant (i.e. stationary):

$$\lambda(B,j+1) = \int Q(x,B,j)\lambda(x,j)dx \text{ for } j = 1,2,... \qquad (23)$$

$$\lambda(B,j) = \begin{cases} \pi(s)/T \text{ if } \{s\} \times 0 \in B \\ \quad 0 \text{ otherwise} \end{cases} \qquad (24)$$

for all B.

1.3 Fundamental questions about the stationary distribution

The fundamental questions related to the stationary distribution are the following. First, does it exist? Second, is it unique? Third, is it stable? Clearly, if the stationary distribution does not exist, we cannot rely on the above definition of the stationary equilibrium. If a stationary distribution exists but is not unique, then the equilibrium depends on initial conditions, which may hard to pin down. Finally, it would be good if we knew that the distribution was (globally) stable, because then we would know that we can find the stationary distribution by iteration starting from any initial distribution.

Existence, uniqueness and global stability have been proved for specific models. However, these properties do not hold generally in incomplete market models. In more complicated models, even the existence of a stationary equilibrium can sometimes only be "proved" numerically.

1.4 Revisiting the optimal capital tax

Recall the Chamley-Judd result from lecture notes 2: In the deterministic neoclassical growth model, the optimal long run capital tax rate is zero. That result was derived in a complete markets set-up. Interestingly, Aiyagari (1995) showed that the result does not hold in incomplete markets models.

To see the intuition behind Aiyagari's result, consider again the infinite horizon savings model discussed above. How would the allocation look if we had a social planner dictating households' choices? Clearly, the social planner would insure households perfectly against the income shocks that they face in the competitive equilibrium. This is because there is no aggregate uncertainty and because individuals suffer from being exposed to uncertainty. Then the Euler equation related to the first-best allocation tells us that in steady state, we must have $\beta(1 + f_k - \delta) = 1$. However, in lecture notes 4, we showed that if $\beta(1+r) = 1$ the households are going to accumulate assets over any finite bound. That would obviously bring the equilibrium interest rate down. Therefore, in the general competitive equilibrium, we must have $\beta(1 + r) < 1$. In other words, in the competitive equilibrium of the incomplete markets model, there is overaccumulation of capital compared to the first-best. A government that can only rely on distortionary taxes, should counteract the tendency to overaccumulate assets by setting a strictly positive capital tax rate.

1.5 Finding the stationary equilibrium

Let us again consider the infinite horizon savings model (sometimes referred to as the Aiyagari model) to illustrate how to find the stationary equilibrium in practice. The steps are the following:

0. Specify all functional forms and parameter values and guess the aggregate capital stock K

1. Compute prices r and w from (6) and (7). Make sure that $\beta(1+r) < 1$. (If $\beta(1+r) \geq 1$, the capital stock is too small.)

2. Solve the individual problem to find the optimal policy function $g(a, s)$.

3. Find the stationary distribution associated with the optimal policy function and the transition probability matrix P. The most straightforward way is by Monte Carlo simulation. Take, say, 10 000 households, and take random draws which determine a sequence of income shocks for, say 1000 periods, for each household. Then simulate each household's savings. Compute the aggregate asset holdings from the simulated distribution in period 1000. You should check that the distribution indeed remains the same up to some tiny change that is due to the fact that the law of large numbers does not hold exactly. (A more efficient and accurate way of finding the stationary distribution is to model the distribution by a step density function.)

4. Update the guess for the aggregate capital stock based on the average asset holdings in the simulated distribution.

5. Iterate 1-4 until the guessed capital stock is the same as the average asset holdings in the simulated distribution.

1.6 Some applications

Early contributions include Aiyagari (1994), Hugget (1993), and Bewley (1992). See LS chapter 17 for references and discussion about these papers.

There is a large and growing literature that applies this type of incomplete markets models to study various government tax-and-transfer programs. The important feature of these models is that redistributive tax-and-transfers program can improve welfare by providing insurance against idiosyncratic risks. See, for instance, Imrohoroglu et al. (1995), Floden and Lindé (2001) and Conesa et al. (2009). Imrohoroglu et al. were the first to use a numerical OLG-model with idiosyncratic uncertainty to analyze social security. The latter two papers focused on labor and capital income taxation. Conesa et al. also solve for the transitionary (aggregate) dynamics.

Another important topic that these models have been used to analyze is income and wealth inequality. See, for instance, Castañeda et al. (2003) and the references therein.

References

[1] Aiyagari (1995), Optimal capital income taxation with incomplete markets, borrowing constraints, and constant discounting, *Journal of Political Economy,* 103(6), 1158-75.

[2] Castañeda, A. Dìaz-Gimenez, J. and Ríos-Rull, J. (2003) Accounting for earnings and wealth inequality, *Journal of Political Economy*, 111 (4), 818-857.

[3] Conesa, J., S. Kitao, D. Krueger (2009) Taxing Capital? Not a Bad Idea after All! *American Economic Review*, 99(1), 25-48.

[4] Floden, M. and J. Lindé (2001), Idiosyncratic Risk in the United States and Sweden: Is There a role for Government Insurance? *Review of Economic Dynamics* 4(2).

[5] Imrohoroglu A. ,S. Imrohoroglu, D. Joines (1995), A Life Cycle Analysis of Social Security, *Economic Theory* 6(1), 83-114.